Homework #1. Due Thursday, September 1st, in class

Important reminder: Only problems (or their parts) marked with the **Q** (quiz) symbol need to be sumitted in writing. You are NOT allowed to discuss those "**Q** questions" with others or use any resources (including web) except for the book and your class noted. ALL assertions must be proved unless explicitly stated otherwise.

Reading:

1. For this homework assignment: Sections 1.1 - 1.6 and Appendix on fields.
2. Before the class on Tue, Aug 30th: Section 1.6. Before the class on Thu, Sep 1st: Section 1.7.

Problems:

Problem 1: Let F be a field, $n \geq 1$ and $Mat_n(F)$ the set of all $n \times n$ matrices with entries in F. Given $A = (a_{ij}) \in Mat_n(F)$, the *trace of* A, denoted by $\mathrm{tr}(A)$, is the sum of all diagonal entries of A:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

(a) Prove that $\mathrm{tr}(\lambda A + \mu B) = \lambda\,\mathrm{tr}(A) + \mu\,\mathrm{tr}(B)$ for all $A, B \in Mat_n(F)$ and $\lambda, \mu \in F$.

Q(b) Let $\mathfrak{sl}_n(F) = \{A \in Mat_n(F) : \mathrm{tr}(A) = 0\}$. Prove that $\mathfrak{sl}_n(F)$ is a subspace of $Mat_n(F)$ (where $Mat_n(F)$ is considered as a vector space over F with respect to the usual matrix addition and scalar multiplication).

Problem 2: Let V be a vector space over a field F. Let W_1 and W_2 be subspaces of V and S_1 and S_2 subsets of V.

(a) Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(b) Prove that $W_1 + W_2$ is a subspace of V, where by definition $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$.

Q(c) Prove that $W_1 + W_2$ is the smallest subspace of V containing W_1 and W_2, that is, if W is any subspace of V containing W_1 and W_2, then W contains $W_1 + W_2$.

Q(d) Prove that $Span(S_1 \cup S_2) = Span(S_1) + Span(S_2)$ in two different ways: first directly from definition of the span and then by using the characterization of spans in Theorem 2.1(b) from Lecture 2 (= Theorem 1.5(b) from the book) and part (c) of this problem.

Problem 3: Let $V = \mathbb{C}$ (complex numbers). Similarly to a discussion in Lecture 1, we can consider V as a vector space over itself ($F = \mathbb{C}$), over reals ($F = \mathbb{R}$) or over rationals ($F = \mathbb{Q}$). Thus, we have three different notions of a subspace of V – to distinguish between them we will use the terms $\mathbb{C}$-subspace, $\mathbb{R}$-subspace and $\mathbb{Q}$-subspace.

(a) Prove that the only $\mathbb{C}$-subspaces of V are V itself and $\{0\}$.

Q(b) Describe all $\mathbb{R}$-subspaces of V (and prove that there are no other subspaces). You may use any results from the book in Sections 1.1-1.6, but state your references clearly.

(c) For every $n \in \mathbb{Z}_{\geq 0}$ construct an explicit example of a $\mathbb{Q}$-subspace W_n which has dimension n (as a vector space over $\mathbb{Q}$). If you did not take 5652, it may not be easy to prove formally that your W_n has dimension n.

Problem 4: Let V be a vector space over a field F and $S = \{v_1, v_2\}$ a subset of V containing precisely two elements. Prove that S is linearly dependent $\iff$ v_1 is a scalar multiple of v_2 or v_2 is a scalar multiple of v_1. (Recall that w is a scalar multiple of v if $w = \lambda v$ for some $\lambda \in F$).

Problem 5: Recall that for an integer $m \geq 2$ we denote by $\mathbb{Z}_m$ the ring of congruence classes mod m and that $\mathbb{Z}_m$ is a field $\iff$ m is a prime.

(a) Let p be a prime, $n \geq 1$ an integer, and let $V = \mathbb{Z}_p^n$, the standard n-dimensional vector space over $\mathbb{Z}_p$. How many elements does V have?

(b) Find all ordered bases for $\mathbb{Z}_2^2$. How many unordered bases does it have? (An unordered basis is a basis in the usual sense; an ordered basis is a basis with a chosen order on its elements).

(c) Now let p be any prime. How many ordered bases does $\mathbb{Z}_p^2$ have? **Hint:** Use Problem 4.

Problem 6: Let S be a linearly dependent subset of a vector space V. Prove that some finite subset of S is linearly dependent.

Problem 7: **Q** Let F be a field and $\mathcal{P}(F)$ the vector space of all polynomials with coefficients in F. Let S be a subset (possibly infinite) of $\mathcal{P}(F)$ such that $0 \notin S$ and any two elements of S have distinct degrees. Prove that S is linearly independent.