Homework #9. Due Thursday, November 20th, in class

Reading:

1. For this homework assignment: Section 25.1 from Kolmogorov-Fomin (class handout) + class notes (Lectures 21-22); alternative reference is Rudin, Section 11.2 (construction of the Lebesgue measure).

2. For next week's classes: I will be distributing another handout from Kolmogorov-Fomin in class next Tuesday, but I recommend that you read Sections 11.4 (measurable functions), 11.5 (simple functions) and beginning of 11.6 (integration) which covers roughly the same material.

Problems:

0. Let A_1, A_2, B_1 and B_2 be subsets of the same set, and assume that $A_1 \cap A_2 = \emptyset$. Prove that

$$B_1 \cap B_2 \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2).$$

(Recall that $X \triangle Y = (X \setminus Y) \sqcup (Y \setminus X)$). Also recall that the result of this problem was used in the proof of Theorem 22.4 in class.

1. Let A be an elementary subset of $\mathbb{R}$ (recall that in class we defined an elementary set as a finite union of (finite) intervals).

(a) Prove that A is a finite disjoint union of intervals.

(b) Recall that we defined $m(A)$ in class as follows: write $A = \sqcup_{k=1}^{n} I_k$, where I_k are pairwise disjoint intervals, and set $m(A) = \sum_{k=1}^{n} \text{length}(I_k)$. Prove that $m(A)$ does not depend on the choice of the decomposition $A = \sqcup_{k=1}^{n} I_k$. **Hint:** This is essentially explained in Kolmogorov-Fomin, but they skip the "basic" case when A itself is an interval.

2. Problem 1 from Kolmogorov-Fomin (p. 268). Note that this is a problem about subsets of $\mathbb{R}^2$ (not $\mathbb{R}$). **Hint for (a):** Show that any open subset of $\mathbb{R}^2$ can be written as a union of squares whose endpoints have rational coordinates.

3.

(a) Let $E = [0,1]$, and let A be a countable subset of E. Prove that A has measure zero (that is, A is measurable and $\mu(A) = 0$).

(b) Prove that the Cantor set C has measure 0.

4. The goal of this problem is to give an example of a non-measurable subset of $[0,1]$. You will need to use the fact that the Lebesque measure μ is countable additive (Theorem 10, p.265 in Rudin).

(a) Define the relation $\sim$ on $[0,1]$ by $x \sim y \iff y - x \in \mathbb{Q}$. Prove that $\sim$ is an equivalence relation.

(b) Let S be a subset of $[0, \frac{1}{3}]$ which contains precisely one element form each equivalence class with respect to $\sim$. Prove that S is non-measurable as follows. Let T be the set of rational numbers in $[0, \frac{2}{3}]$, and for each $t \in T$ let $S_t = \{s + t : s \in S\}$. Show that the sets $\{S_t : t \in T\}$ are pairwise disjoint and their union $\sqcup_{t \in T} S_t$ contains $[\frac{1}{3}, \frac{2}{3}]$ and is contained in $[0,1]$. Assuming that S is measurable, deduce that each S_t is measurable and $\mu(S_t) = \mu(S)$ for each t, and then reach a contradiction with countable additivity of μ.