Homework #8. Due Thursday, November 6th, in class

Reading:

1. For this homework assignment: Chapter 7 + class notes (Lectures 14-18)

2. For next week's classes: Section 7.7 (Stone-Weierstrass Theorem).

Problems:

1. Let $a, b \in \mathbb{R}$ with $a < b$, and let $\{f_n\}$ be a sequence of differentiable functions from $[a, b]$ to $\mathbb{R}$. Suppose that both the sequences $\{f_n\}$ and $\{f_n'\}$ are uniformly bounded. Prove that $\{f_n\}$ has a subsequence which converges uniformly on $[a, b]$.

2. Let X be a compact metric space, $(C(X), d_\infty)$ the space of continuous functions from X to $\mathbb{R}$ with uniform metric d_∞ (given by $d_\infty(f, g) = \max_{x \in X} |f(x) - g(x)|$). Prove that a subset $\mathcal{F}$ of $C(X)$ is compact (with respect to d_∞) $\iff$ $\mathcal{F}$ satisfies the following three conditions:

(i) $\mathcal{F}$ is uniformly closed, that is, $\mathcal{F}$ is closed with respect to d_∞
(ii) $\mathcal{F}$ is uniformly bounded
(iii) $\mathcal{F}$ is equicontinuous

Hint: For the forward direction, the main thing to prove is that $\mathcal{F}$ is equicontinuous. Assuming the contrary, show that $\mathcal{F}$ contains a sequence with no equicontinuous subsequence and then use Theorem 7.24 from Rudin. For the backwards direction combine Arzela-Ascoli Theorem with the fact that $(C(X), d_\infty)$ is a complete metric space (Theorem 7.15 in Rudin).

3. The goal of this problem is to show that the statement of Arzela-Ascoli Theorem need not hold for sequences of continuous functions from X to $\mathbb{R}$ if X is not compact.

(a) Consider functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \begin{cases} \frac{|x|}{n} & \text{if } |x| \leq n \\ 1 & \text{if } |x| > n \end{cases}$
Prove that the sequence $\{f_n\}$ is uniformly bounded and equicontinuous, but does not have a uniformly convergent subsequence. Deduce that Arzela-Ascoli Theorem does not hold for $X = \mathbb{R}$.

(b) (bonus) Now let (X, d) be any unbounded metric space. Show that there exists a sequence of continuous functions $f_n : X \to \mathbb{R}$ which is uniformly bounded and equicontinuous, but does not have a uniformly convergent subsequence. **Hint:** You can construct such a sequence using functions of the form $f(x) = d(x, a)$ (for a fixed $a \in X$).

4. Problem 7.3:15 from Bergman's supplement (page 79), see
`http://math.berkeley.edu/~gbergman/ug.hndts/m104_Rudin_exs.pdf`

Hint: Start by explicitly describing open balls of radius < 1 in X.

5.

(a) Prove that the (direct) analogue of Weierstrass Theorem does not hold for $C(\mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$: Show that there exists $f \in C(\mathbb{R})$ which cannot be uniformly approximated by polynomials, that is, there is no sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $\mathbb{R}$. **Hint:** Use the fact that any non-constant polynomial $p(x)$ tends to $\pm\infty$ as $x \to \infty$.

(b) Now prove that the following (weak) version of Weierstrass Theorem holds for $C(\mathbb{R})$: for any $f \in C(\mathbb{R})$ there exists a sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $[a, b]$ for any closed interval $[a, b]$ (of course, the point is that a single sequence will work for all intervals). **Hint:** It is enough to do this for intervals of the form $[-k, k]$ for $k \in \mathbb{N}$ (why?). To construct a sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $[-k, k]$ for each k, apply Weierstrass theorem on each interval and then use a diagonal-type argument.

6. Let $a < b$ be real numbers and let $\mathcal{P}_{even}[a, b] \subseteq C[a, b]$ be the set of all even polynomials (that is, polynomials which only involve even powers of x).

(a) Use Stone-Weierstrass Theorem to prove that $\mathcal{P}_{even}[a, b]$ is dense in $C[a, b] \iff ab \geq 0$ (that is, $0 \leq a < b$ or $a < b \leq 0$).

(b) (optional) Now prove the "$\Leftarrow$" direction in (b) using only Weiertrass Theorem (but not Stone-Weierstrass Theorem). **Hint:** WOLOG assume that $0 \leq a < b$. Start by showing that any continuous function in $g \in C[a, b]$ can be written as $g(x) = h(x^2)$ for some continuous function $h \in C[a^2, b^2]$.