Homework #6. Due Thursday, October 16th, in class

Reading:

1. For this homework assignment: 4.1-4.4, 5.2 + class notes (Lectures 10-12)

2. For next week's class: Sections 7.1-7.3 (discussion of main problems, uniform convergence, uniform convergence and continuity).

Problems:

1. Let (X, d) be a metric space. Recall that given a point $x \in X$ and a subset Z of X, we define $d(x, Z)$ (the distance from x to Z) by

$$d(x, Z) = \inf\{d(x, z) : z \in Z\}.$$

(a) Prove that $d(x, Z) \geq d(y, Z) - d(x, y)$ for all $x, y \in X$ and $Z \subseteq X$

(b) Now fix $Z \subseteq X$, and define $d_Z : X \to \mathbb{R}$ by $d_Z(x) = d(x, Z)$. Use (a) to prove that d_Z is uniformly continuous.

Recall that in class we showed that for a fixed $a \in X$, the function $d_a : X \to \mathbb{R}$ given by $d_a(x) = d(x, a)$ is uniformly continuous.

2. Again let (X, d) be a metric space. Let $a \in X$ and K a compact subset of X. Prove that there exists $k \in K$ such that $d(a, k) = d(a, K)$ (this is equivalent to saying that the set $\{d(a, z) : z \in K\}$ has the minimal element). **Hint:** Apply Theorem 4.16 from Rudin to a suitable function.

3. Let X be a metric space. Prove that X is disconnected if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $f(X) = \{1, -1\}$. **Hint:** The "$\Leftarrow$" direction is easy. For the "$\Rightarrow$" direction, assume that $X = A \sqcup B$ with A, B closed, and show that the function $f : X \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \in B \end{cases}$$

is continuous.

4. This problem describes a fancy way to show that closed intervals in $\mathbb{R}$ are connected. We will say that a metric space (X, d) has the *chain property* if for any $x, y \in X$ and $\delta > 0$ there exists a finite sequence $x_0, x_1, \ldots, x_n$ of points of X such that $x_0 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \delta$ for all i.

(a) Let X be a compact metric space with the chain property. Prove that X is connected. **Hint:** Assume that X is disconnected, and use Problem 3 and uniform continuity to reach a contradiction.

(b) Prove that a finite closed interval $[a, b] \subseteq \mathbb{R}$ has the chain property and deduce from (a) that $[a, b]$ is connected.

5.

(a) Let X be a disconnected metric space, so that $X = A \sqcup B$ for some non-empty closed subsets A and B. Prove that if C is any connected subset of X, then $C \subseteq A$ or $C \subseteq B$.

(b) A metric space X is called *path-connected* if for any $x, y \in X$ there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$ (informally, this means that any two points in X can be joined by a path in X). Use (a) and Theorem 4.22 from Rudin (=Theorem 11.1 from class) to prove that any path-connected metric space is connected.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

(a) Assume that f' is bounded, that is, there exists $M \in \mathbb{R}$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.

(b) Now assume that $f'(x) \to \infty$ as $x \to \infty$. Prove that f is not uniformly continuous.

Hint: Use the mean-value theorem.

7 (bonus). Prove that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is rational whenever x is irrational and vice versa $f(x)$ is irrational whenever x is rational.