**Homework #5. Due Thursday, October 2nd, in class**

**Reading:**

1. For this homework assignment: Sections 2.5, 4.1-4.4 + class notes (Lectures 9 and 10)

2. For next week's classes: Sections 4.4-4.6 (continuity and connectedness, discontinuities, monotonic functions).

**Problems:**

**1.** Complete the proof of Theorem 10.3 started in class. **Theorem 10.3** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and assume that $X$ is compact. Then any continuous function $f : X \to Y$ is uniformly continuous.* **Hint:** Suppose that $f : X \to Y$ is continuous, but not uniformly continuous. Then, as we discussed in class, there exist $\varepsilon > 0$ and sequences $\{a_n\}$ and $\{b_n\}$ in $X$ such that $d_X(a_n, b_n) < \frac{1}{n}$, but $d_Y(f(a_n), f(b_n)) \geq \varepsilon$ for all $n$. Since $X$ is compact, it is sequentially compact, so there exists a subsequence $\{a_{n_k}\}$ which converges to some $a \in X$. Use the fact that $d_X(a_n, b_n) \to 0$ as $n \to \infty$ to deduce that the sequence $\{b_{n_k}\}$ converges to $a$ as well. Now use Theorem 9.2 (characterization of continuity in terms of convergent sequences) to reach a contradiction with the assumption that $d_Y(f(a_n), f(b_n)) \geq \varepsilon$ for all $n$.

**2.** Let $X$ and $Y$ be metric spaces.

(a) Let $U$ be an open subset of $X$. Let $f : X \to Y$ and $g : X \to Y$ be functions such that $f(x) = g(x)$ for all $x \in U$ and $f$ is continuous at every point of $U$. Prove that $g$ is continuous at every point of $U$ as well.

(b) Give an example showing that the assertion of (a) may be false if we do not assume that $U$ is open.

**3.**

(a) Let $X$ be any set with discrete metric ($d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$). Prove that for any metric space $Y$, any function $f : X \to Y$ is continuous.

(b) Use (a) to show that there exist metric spaces $X$ and $Y$ and a function $f : X \to Y$ such that $f$ is continuous and bijective, but $f^{-1} : Y \to X$ is not continuous (recall that we proved in class that this cannot happen if $X$ is compact). **Hint:** You can construct an example where $X = Y$ as sets (but with different metrics) and $f : X \to Y$ is the identity function ($f(x) = x$ for all $x$).

**Terminology/Notation.** Let $A$ be a set and $\{A_\alpha\}$ a collection of subsets of $A$. We say that $A$ is a **disjoint union** of $\{A_\alpha\}$ if $A = \cup A_\alpha$ and $A_\alpha \cap A_\beta = \emptyset$ for any $\alpha \neq \beta$. If $A$ is a disjoint union of $\{A_\alpha\}$, we write $A = \sqcup A_\alpha$.

**4.** Let $X$ be a metric space. Prove that the following conditions are equivalent:

(i) There exist non-empty closed subsets $A$ and $B$ of $X$ such that $X = A \sqcup B$ (disjoint union of $A$ and $B$)

(ii) There exist non-empty open subsets $A$ and $B$ of $X$ such that $X = A \sqcup B$

(iii) There exist non-empty subsets $C$ and $D$ of $X$ such that $X = C \sqcup D$ and in addition $C \cap \overline{D} = D \cap \overline{C} = \emptyset$

A metric space $X$ is called *disconnected* if it satisfies either of these conditions ((iii) is essentially the definition from Rudin, but (i) or (ii) are more standard). $X$ is called *connected* if it is not disconnected.

**5.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric space, and consider the product space $X \times Y$ with metric $d$ defined in HW#4.1.

(a) Prove that for every $x \in X$, the set $\{x\} \times Y = \{(x, y) : y \in Y\}$ is closed in $X \times Y$. Similarly, prove that for every $y \in Y$, the set $X \times \{y\}$ is closed in $X \times Y$.

(b) Prove that the metric spaces $Y$ and $\{x\} \times Y$ (where $x$ is a fixed element of $X$) are isometric (see HW#4 for the definition of being isometric). Deduce that if $Y$ is connected, then $\{x\} \times Y$ is connected. Likewise, if $X$ is connected, then $X \times \{y\}$ is connected (there is no need to write down the proof of the latter statement).

(c) Now assume that $X$ and $Y$ are both connected. Prove that $X \times Y$ is also connected. **Hint:** Assume that $X \times Y$ is disconnected, so $X \times Y = A \sqcup B$ for some non-empty closed subsets $A$ and $B$ of $X \times Y$. Prove that for each $x \in X$, the intersection $(\{x\} \times Y) \cap A$ is either $\{x\} \times Y$ or $\emptyset$. Similarly, prove that for each $y \in Y$, the intersection $(X \times \{y\}) \cap A$ is either $X \times \{y\}$ or $\emptyset$. Deduce that this is possible only if $A = \emptyset$ or $A = X \times Y$ (in which case $B = \emptyset$), thus reaching a contradiction. **Hint:** Draw a picture in the case where $X = Y = [0, 1]$ (so that $X \times Y$ is a unit square).

(d) Now prove that subsets of $\mathbb{R}^2$ of the form $(a, b) \times (c, d)$ and $[a, b] \times [c, d]$ are connected.

**6.** The goal of this problem is to prove that any open subset of $\mathbb{R}$ (with standard metric) is a **disjoint** union of countably many open intervals (recall that by our new convention, countable includes finite).

So, let $U$ be any open subset of $\mathbb{R}$.

(a) Define the relation $\sim$ on $U$ by setting $x \sim y \iff x = y$ or ($x < y$ and $[x, y] \subset U$) or ($y < x$ and $[y, x] \subset U$). Prove that $\sim$ is an equivalence relation.

(b) Let $A$ be an equivalence class with respect to $U$. Show that $A$ is an open interval. **Hint:** Consider four cases: $A$ is bounded from above and below; $A$ is bounded from above but not from below etc.. In the first case show that $A = (\inf A, \sup A)$; in the second case show that $A = (-\infty, \sup A)$ etc.

(c) Deduce from (b) that $U$ is a disjoint union of intervals. Then prove that the number of those intervals is countable. **Hint:** There are countably many rational numbers.

**7.** Use Problem 5 to show that the analogue of Problem 6 does not hold in $\mathbb{R}^2$, that is, there exist open subsets of $\mathbb{R}^2$ which are not representable as disjoint unions of open discs (an open disc is an open ball in $\mathbb{R}^2$).

**8.** Let $X$ be any metric space, $\{x_n\}_{n\in\mathbb{N}}$ a convergent sequence in $X$ and $x = \lim_{n\to\infty} x_n$. Prove that the set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact directly from the definition of compactness (do not use sequential compactness).