Homework #4. Due Thursday, September 25th, in class

Reading:

1. For this homework assignment: Sections 2.4, 2.5 + class notes (Lectures 7 and 8)

2. For next week's classes: carefully read Theorem 2.40 before Tuesday's class. We will prove Theorem 8.3 on Tuesday using essentially the same argument. Note that Theorem 8.3 is a special case of Theorem 2.40 (with $k=1$); however, the general case of Theorem 2.40 follows from Theorem 8.3 and Problem 1 in this assignment. Also read sections 4.1-4.4 (limits of functions, continuous functions, continuity and compactness, continuity and connectedness).

Problems:

1. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $X = X_1 \times X_2$, and define the function $d : X \times X \to \mathbb{R}_{\geq 0}$ by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

(a) Prove that (X, d) is a metric space.

(b) Assume that (X_1, d_1) and (X_2, d_2) are both sequentially compact. Prove that (X, d) is also sequentially compact.

Hint for (b): Take any sequence in X – it has the form $\{(a_n, b_n)\}$ for some $a_n \in X_1$ and $b_n \in X_2$. Since X_1 is sequentially compact, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. Now look at the sequence $\{b_{n_k}\}$. Since X_2 is sequentially compact, it has a convergent subsequence $\{b_{n_{k_s}}\}_{s\in\mathbb{N}}$. Now prove that the subsequence $\{(a_{n_{k_s}}, b_{n_{k_s}})\}$ of $\{(a_n, b_n)\}$ converges.

2. Let X be metric space, and let $Z \subset Y$ be subsets of X. Prove that Z is closed as a subset of $Y \iff Z = Y \cap K$ for some closed subset K of X. Deduce that if Z is closed in X, then Z is closed in Y. **Note:** The corresponding result with closed replaced by open is also valid and appears as Theorem 2.30 in Rudin.

3. Let X be a metric space. Prove that X is compact $\iff$ X satisfies the following property:

> Let $\{K_\alpha\}$ be any collection of closed subsets of X such that for any finite subcollection $K_{\alpha_1}, \ldots, K_{\alpha_n}$, the intersection $K_{\alpha_1} \cap \ldots \cap K_{\alpha_n}$ is non-empty. Then the intersection of all sets in $\{K_\alpha\}$ is non-empty.

Hint: If you have a collection of closed subsets with empty intersection, how do you get an open cover out of that collection?

4. Let (X,d) be a metric space, with $X \neq \emptyset$. Prove that the following three conditions are equivalent (as defined in class, (X,d) is called *bounded* if it satisfies either of those conditions):

(i) There exists $x \in X$ and $M \in \mathbb{R}$ such that $N_M(x) = X$.
(ii) For any $x \in X$ there exists $M \in \mathbb{R}$ such that $N_M(x) = X$.
(iii) The set $\{d(x,y) : x, y \in X\}$ is bounded above as a subset of $\mathbb{R}$.

5. Suppose that A is a compact connected subset of $\mathbb{R}$. Prove that A is a closed interval, that is, $A = [a,b]$ for some $a \le b$. **Hint:** As proved in class, compact subsets are always closed and bounded, so $\sup(A)$ and $\inf(A)$ both exist (in $\mathbb{R}$). Use the fact that A is closed to prove that $\sup(A), \inf(A) \in A$ and then use Theorem 2.47 in Rudin to prove that $A = [\inf(A), \sup(A)]$.

6. This problem introduces the concept of the **completion** of a metric space. In the statement of the problem we will use the notion of isometric metric spaces. Metric spaces (X,d) and (X',d') are called isometric (to each other) if there is a bijection $f : X \to X'$ such that $d(x,y) = d'(f(x), f(y))$ for all $x, y \in X$.

Let (X,d) be a metric space. Let Ω be the set of all Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X$ for each n. Define the relation $\sim$ on Ω by setting

$$\{x_n\} \sim \{y_n\} \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

(a) Prove that $\sim$ is an equivalence relation.

Now let $\overline{X} = \Omega/\sim$, the set of equivalence classes with respect to $\sim$. The equivalence class of a sequence $\{x_n\}$ will be denoted by $[x_n]$. For instance, $[\frac{1}{n}] = [\frac{1}{n^2}]$ since the sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$ are equivalent. Given an element $x \in X$, we will denote by $[x] \in \overline{X}$ the equivalence class of the constant sequence all of whose elements are equal to x.

Now define the function $D : \overline{X} \times \overline{X} \to \mathbb{R}_{\ge 0}$ by setting

$$D([x_n],[y_n]) = \lim_{n\to\infty} d(x_n, y_n) \qquad (***)$$

(b) Prove that the limit on the right-hand side of (***) always exists and that the function D is well-defined (that is, if $[x_n] = [x'_n]$ and $[y_n] = [y'_n]$, then $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$). **Hint:** For the existence of the limit use the fact that $\mathbb{R}$ is a complete metric space.
(c) Prove that $(\overline{X}, D)$ is a metric space
(d) Consider the map $\iota : X \to \overline{X}$ given by $\iota(x) = [x]$ (that is, ι sends each x to the equivalence class of the corresponding constant sequence).

Prove that ι is injective and $D(\iota(x), \iota(y)) = d(x, y)$ for all $x, y \in X$. This implies that (X, d) is isometric to the metric space $(\iota(X), D)$ (so identifying X with $\iota(X)$, we can think of X as a subset of $(\overline{X}, D)$).

(e) (bonus) Now prove that $(\overline{X}, D)$ is a complete metric space. For this reason, $(\overline{X}, D)$ is called the *completion* of (X, d). **Hint:** Let $\{f_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\overline{X}, D)$. By definition of $(\overline{X}, D)$, each f_k is an equivalence of a Cauchy sequence in X, that is $f_k = [x_{k,n}]$ for some elements $x_{k,n} \in X$, $n \in \mathbb{N}$. Prove that there is an increasing sequence of natural numbers $m_1 < m_2 < \ldots$ such that if we define $y_n = x_{n,m_n}$, then the sequence $\{y_n\}$ is Cauchy and its equivalence class $[y_n]$ is the limit of the sequence $\{f_k\}$ (in $(\overline{X}, D)$).

(f) (bonus) Assume that $X = \mathbb{Q}$ (rationals) and $d(x, y) = |x-y|$. Prove that the completion of (X, d) is isometric to reals (with the standard metric).

7. A metric space (X, d) is called **ultrametric** if for any $x, y, z \in X$ the following inequality holds:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

(Note that this inequality is much stronger than the triangle inequality). If X is any set and we define the metric d on X by $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$, then clearly (X, d) is ultrametric. A more interesting example of an ultrametric space is given in the next problem.

Prove that properties (i) and (ii) below hold in any ultrametric space (X, d) (note that both properties are counter-intuitive since they are very far from being true in $\mathbb{R}$).

(i) Take any $x \in X$, $\varepsilon > 0$ and take any $y \in N_\varepsilon(x)$. Then $N_\varepsilon(y) = N_\varepsilon(x)$. This means that if we take an open ball of fixed radius around some point x, then for any other point y from that open ball, the open ball of the same radius, but now centered at y, coincides with the original ball. In other words, any point of an open ball happens to be its center.

(ii) Prove that a sequence $\{x_n\}$ in X is Cauchy $\iff$ for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $d(x_{n+1}, x_n) < \varepsilon$ for all $n \geq M$. **Note:** The forward implication holds in any metric space.

(iii) Give an example showing that condition (ii) fails for $X = \mathbb{R}$.

8. Let p be a fixed prime number. Define the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ as follows: given a nonzero $x \in \mathbb{Q}$, we can write $x = p^a \frac{c}{d}$ for some $a, c, d \in \mathbb{Z}$ where c and d are not divisible by p . Define $|x|_p = p^{-a}$ (note that the above representation is not unique, but it is easy to see that a is uniquely determined by x). For instance,

$$\left|\frac{9}{20}\right|_p = \begin{cases} \frac{1}{9} & \text{if } p = 3 \\ 4 & \text{if } p = 2 \\ 5 & \text{if } p = 5 \\ 1 & \text{for any other } p. \end{cases}$$

Also define $|0|_p = 0$. Now define the function $d_p : \mathbb{Q}\times\mathbb{Q} \to \mathbb{R}_{\geq 0}$ by $d_p(x, y) = |y - x|_p$.

(a) Prove that $(\mathbb{Q}, d_p)$ is an ultrametric space. (Note: the completion of this metric space is usually denoted by $\mathbb{Q}_p$ is called *p-adic numbers*).

(b) Describe explicitly the set $N_1(0)$ (the open ball of radius 1 centered at 0) in $(\mathbb{Q}, d_p)$.

(c) Let d be the standard metric on $\mathbb{Q}$ (that is, $d(x, y) = |y - x|$ where $|\cdot|$ is the usual absolute value). Give examples of sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{Q}$ such that

(i) $x_n \to 0$ in $(\mathbb{Q}, d_p)$ but $\{x_n\}$ is unbounded as a sequence in $(\mathbb{Q}, d)$

(ii) $y_n \to 0$ in $(\mathbb{Q}, d)$ but $\{y_n\}$ is unbounded as a sequence in $(\mathbb{Q}, d_p)$

Note: It is true that in any metric space Cauchy (in particular, convergent) sequences are bounded.