Homework #3. Due Thursday, September 18th, in class

Reading:

1. For this homework assignment: Sections 2.2, 3.1-3.3 + class notes

2. For next week classes: 2.3 (compact sets), read about sequential compactness vs compactness at

`http://math.berkeley.edu/~jdahl/141/141sc.pdf`;

also read 3.1-3.3 again, this time including the results dealing with compactness.

Try to at least seriously think about Problems 1-5 before Tuesday as they will be relevant to the material we will be discussing in class.

Problems:

1. Given a metric space (X, d), a point $x \in X$ and $\varepsilon > 0$, define $B_\varepsilon(x) = \{y \in X : d(y, x) \leq \varepsilon\}$, called the *closed ball of radius ε centered at x.*

(a) Prove that $B_\varepsilon(x)$ is always a closed subset of X.

(b) Deduce from (a) that $\overline{N_\varepsilon(x)} \subseteq B_\varepsilon(x)$, that is, the closure of the open ball of radius ε centered at x is contained in the respective closed ball.

(c) Is it always true that $\overline{N_\varepsilon(x)} = B_\varepsilon(x)$? Prove or give a counterexample.

2. Let (X, d) be a metric space and Y a subset of X.

(a) (practice) Suppose that X is sequentially compact and Y is a closed subset of X. Prove that Y is also sequentially compact. (We will prove this result in class on Tue, Sep 16, but I recommend that you try to do it yourself – the proof is short and simple).

(b) Now assume that Y is sequentially compact. Prove that Y is closed in X (we are not assuming anything about X).

Note: Since sequential compactness is equivalent to compactness for metric spaces, as we will prove in class, the results of parts (a) and (b) of this problem are equivalent to Theorems 2.35 and 2.34 in Rudin, respectively. The point of this exercise is to prove the results working directly with the definition of sequential compactness.

3. Let $\{x_n\}$ be a sequence in a metric space (X, d), and let x be some element of X. Prove that the following conditions are equivalent:

(i) some subsequence of $\{x_n\}$ converges to x

(ii) for every $\varepsilon > 0$ there are infinitely many n for which $x_n \in N_\varepsilon(x)$.

Let (X, d) be a metric space and $\varepsilon > 0$. A subset S of X is called an *ε-net* if for any $x \in X$ there exists $s \in S$ such that $d(x, s) < \varepsilon$. In other words, S is an ε-net if X is the union of open balls of radius ε centered at elements of S.

4. Let T be a subset of a metric space (X, d). Prove that the following are equivalent:

(i) The closure of T is the entire X;
(ii) $U \cap T \neq \emptyset$ for any non-empty open subset U of X;
(iii) T is an ε-net for every $\varepsilon > 0$.

The subset T is called *dense* (in X) if it satisfies these equivalent conditions.

5. Rudin 2.23.

6. Let X be a set, and let d_1 and d_2 be two different metrics on X. Given $x \in X$ and $\varepsilon > 0$, define $N_\varepsilon^1(x) = \{y \in X : d_1(y, x) < \varepsilon\}$, the ε-neighborhood of x with respect to d_1, and similarly define $N_\varepsilon^2(x) = \{y \in X : d_2(y, x) < \varepsilon\}$. We will say that d_1 and d_2 are *topologically equivalent* if a subset S of X is open with respect to d_1 $\iff$ it is open with respect to d_2. (**Note:** we say that S is open with respect to a metric d on a set X if S is open as a subset of the metric space (X, d)).

(a) Prove that d_1 and d_2 are topologically equivalent if and only if for every $\varepsilon > 0$ and every $x \in X$ there exists $\delta_1, \delta_2 > 0$ (depending on both ε and x) such that $N_{\delta_1}^1(x) \subseteq N_\varepsilon^2(x)$ and $N_{\delta_2}^2(x) \subseteq N_\varepsilon^1(x)$.
(b) Suppose that there exist real numbers $A, B > 0$ such that $d_1(x, y) \leq A d_2(x, y)$ and $d_2(x, y) \leq B d_1(x, y)$ for all $x, y \in X$. Use (a) to prove that d_1 and d_2 are topologically equivalent.
(c) Now use (b) to prove that the Euclidean and Manhattan metrics on $\mathbb{R}^n$ are topologically equivalent.

7. Let $[a, b] \subset \mathbb{R}$ be a closed interval, and let be $X = C[a, b]$ the set of all continuous functions $f : [a, b] \to \mathbb{R}$. Recall that in class we introduced two different metrics on X,

(i) the uniform metric, denoted d_∞, and given by

$$d_\infty(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$$

(ii) the L^1- metric, denoted d_1, and given by $d_1(f, g) = \int_a^b |f(x) - g(x)|\, dx$.

Let $S = \{f \in X : |f(x)| < 1 \text{ for all } x \in [a, b]\}$. Prove that S is open with respect to d_∞, but not open with respect to d_1 (and hence d_1 and d_∞ are not topologically equivalent).

Hint: Let $\mathbf{0} \in X$ denote the function which is identically zero on $[a, b]$. Prove that for any $\varepsilon > 0$ there exists a function f_ε such that $d_1(f_\varepsilon, \mathbf{0}) < \varepsilon$ and $f_\varepsilon(x) \geq 1$ for some $x \in [a, b]$. This implies that S is not open with respect to d_1.

8. Bolzano-Weierstrass theorem implies that closed balls in $\mathbb{R}$ are sequentially compact. Let $X = C[a, b]$ (as defined in Problem 2), considered as a metric space with uniform metric d_∞. Prove that the set $B_1(\mathbf{0})$, the closed ball of radius 1 centered at $\mathbf{0}$ in X, is not sequentially compact (all you need to know about sequential compactness for this problem is the definition).