Homework #2. Due Thursday, September 11th, in class

Reading:

1. For this homework assignment: Chapter 1, 2.1 (finite, countable and uncountable sets)

2. For next week classes: 2.2 (metric spaces), 3.1-3.3 (convergent sequences, subsequences, Cauchy sequences), 2.5(connected sets). Note that 3.3 contains some results on compact spaces (Section 2.3) – you can skip those for now as we will be covering material in a different order, with compactness appearing later.

Problems:

1.

(a) Prove Theorem 1.20 from Rudin directly from the construction of $\mathbb{R}$ as Dedekind cuts (without using Theorem 1.19).

(b) Deduce from Theorem 1.20(a) that if $x \in \mathbb{R}$ is such that $x < \frac{1}{n}$ for all $n \in \mathbb{N}$, then $x \leq 0$.

Note: In the statement of Theorem 1.20(b) the rational number p should be interpreted as the corresponding Dedekind cut $p^* = \{x \in \mathbb{Q} : x < p\}$. **Hint:** In the proof of Theorem 1.20(a) you may want to think of nx as $\underbrace{x + \ldots + x}_{n \text{ times}}$ rather than the product of n and x.

2. Let $\mathbb{R}$ be real numbers thought of as Dedekind cuts. Recall that in class we defined the subset $\mathbb{P}$ of positive real numbers by setting $A \in \mathbb{P} \iff 0^* \subset A$ and then defined the order relation $<$ on $\mathbb{R}$ by setting $A < B \iff B - A \in \mathbb{P}$. Prove that this definition of order is consistent with the one given in Rudin, that is, $A < B \iff A \subset B$ (as subsets of $\mathbb{Q}$).

Definition: Two sets X and Y are said to have the same cardinality if there is a bijection from X to Y.

3. Let A be an uncountable set and B a countable subset of A.

(i) Prove that $A \setminus B$ is uncountable.

(ii) Prove that A and $A \setminus B$ have the same cardinality. **Hint:** Choose any countable subset C of $A \setminus B$ and then use things proved in class to show that the identity map $f : (A \setminus B) \setminus C \to (A \setminus B) \setminus C$ can be extended to a bijection from $A \setminus B$ and A.

4. Let X and Y be any sets, and define X^Y to be the set of all functions $f : Y \to X$. Prove that if $|X| \geq 2$, then Y and X^Y do not have the same cardinality. Note that Theorem 4.6 from class is a special case of this problem where $Y = \mathbb{N}$.

5. Let $I = [0, 1]$, the set of all real numbers between 0 and 1, including 0 and 1. In this problem you can use the following fact with proof: there is a bijection between I and the set of all sequences $a_1, a_2, \ldots$, where each $a_i = 0$ or 1 and 1 appears in the sequence infinitely many times (this bijection comes from binary representation of real numbers).

(i) Use Problem 3 to show that I has the same cardinality as $\{0, 1\}^{\mathbb{N}}$ (the set of all infinite sequences of 0 and 1).

(ii) Use (i) to prove that I has the same cardinality as $I \times I$ (note that geometrically $I \times I$ is just a unit square).

6. Rudin 2.2.

7. Rudin 2.11.

8. Go over the material in Sections 3.4-3.14. Note that while Sections 3.1-3.3 deal with general metric spaces, the remainder of Chapter 3 is mostly about real and complex sequences and series. This material should be mostly familiar to you from 3310 (in fact, you probably saw most of the results, albeit without proofs, already in Calc I/II). For this reason I do not plan to spend much time on these sections in class; however, we will need to use the results from these sections (especially the ones dealing with series) later in the course. Technically this reading assignment does not have to be done this week; however, I encourage you to do it right away, as this week's written assignment is probably one of the shortest this term.