Homework #10. Due Thursday, December 4th, in class

Reading:

1. For this homework assignment: Sections 28.1-28.4 and 29.1 from Kolmogorov-Fomin (make sure to read the definition of Lebesgue integral for non-simple functions which we have not discussed in class yet) + class notes (Lectures 23-24) + section on measurable functions in Rudin. Also review the definition of Riemann integral (pp. 120-121).

2. For the class on Tue, Nov 25: Section 29 from Kolmogorov-Fomin. For the classes on Dec 2,4: TBA.

Problems:

1. Let $D:\mathbb{R}\to\mathbb{R}$ be the Dirichlet function (defined by $D(x)=1$ if $x\in\mathbb{Q}$ and 0 if $x\notin\mathbb{Q}$), and let $a<b$ be real numbers.

(a) Prove that D is Lebesgue-integrable on $[a,b]$ and that $\int\limits_{[a,b]} D\,d\mu=0$.

(b) Prove that D is not Riemann-integrable on $[a,b]$.

2. Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue-integrable function, and let $g:[a,b]\to\mathbb{R}$ be another function which coincides with f almost everywhere (by definition this means that $\mu\{x\in[a,b]:f(x)\neq g(x)\}=0$). Prove that g is also Lebesgue-integrable and $\int\limits_{[a,b]} f\,d\mu=\int\limits_{[a,b]} g\,d\mu$.

Hint: First prove the result in the special case where f and g are simple. For a general case, let $B=\{x\in[a,b]:f(x)\neq g(x)\}$, choose a sequence $\{s_n\}$ of integrable simple functions such that $s_n\rightrightarrows f$ on $[a,b]$ (which exists by Lemma 24.3) and a sequence $\{t_n\}$ of functions with countable image (not necessarily measurable) such that $t_n\rightrightarrows g$ on $[a,b]$ (which exists by the same argument as in the proof of Lemma 24.3). Define functions $s'_n:[a,b]\to\mathbb{R}$ by

$$s'_n(x)=\begin{cases} s_n(x) & \text{if } x\notin B\\ t_n(x) & \text{if } x\in B.\end{cases}$$

Show that the functions s'_n are simple (in particular, measurable), $s'_n\rightrightarrows g$ on $[a,b]$ and deduce the general case from the special case.

3. Let $C\subset[0,1]$ be the Cantor set. The Cantor staircase function is the unique *continuous* function $f:[0,1]\to[0,1]$ which is defined on the complement of the Cantor set by conditions $f(x)=\frac{1}{2}$ if $\frac{1}{3}<x<\frac{2}{3}$, $f(x)=\frac{1}{4}$ if $\frac{1}{9}<x<\frac{2}{9}$, $f(x)=\frac{3}{4}$ if $\frac{7}{9}<x<\frac{8}{9}$, $f(x)=\frac{1}{8}$ if $\frac{1}{27}<x<\frac{2}{27}$, $f(x)=\frac{3}{8}$ if

$\frac{7}{27} < x < \frac{8}{27}$ etc. (Note that this function provides the most natural answer to Problem 2(c) on the second midterm). See

`http://en.wikipedia.org/wiki/Cantor_function`

Prove that $\int\limits_{[0,1]} f\, d\mu = \frac{1}{2}$. **Hint:** Use Problem 2 and Problem 3 from HW#9, to construct a suitable simple function g such that $\int\limits_{[0,1]} f\, d\mu = \int_{[0,1]} g\, d\mu$ and then compute $\int\limits_{[0,1]} g\, d\mu$ directly.

4. Rudin, Problem 3 after Chapter 11 (p. 332). **Hint:** Use Cauchy criterion to express the set in question in terms of sets of the form $\{x : |f_n(x) - f_m(x)| < \frac{1}{k}\}$ using countable unions and countable intersections.

5. Kolmogorov-Fomin, Problem 6 after Section 28 (p.292)

6. Kolmogorov-Fomin, Problem 8 after Section 28 (p.292)

7 (bonus). Kolmogorov-Fomin, Problem 9 after Section 28 (pp.292-293).