Homework #9. Due Thursday, November 2nd

Reading:

1. For this homework assignment: Ben Webster's notes (Lectures 21) and class notes (Lectures 18 and 19).

2. Plan for next week:

   – Next Tuesday we will introduce group algebras, class functions and the regular representation (see Sections 4.3 and 4.4 of Steinberg's book) and start proving Theorem 17.1(1).

   – On Thursday we will finish proving Theorem 17.1(1), derive some consequences and then briefly talk about group actions. Read the first few pages of the following blurb by Keith Conrad, at least if you have not seen group actions before.

`http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/gpaction.pdf`

Problems:

For problems (or their parts) marked with a *, a hint is given later in the assignment. Do not to look at the hint(s) until you seriously tried to solve the problem without it.

1.* Let G be a finite group, H a subgroup of G and V an irreducible complex representation of G. While V need not be irreducible as a representation of H, it is completely reducible, that is, we can write $V = \oplus_{i=1}^k V_i$ where each V_i is an irreducible subrepresentation of H. Prove that

$$\dim(V) \leq \dim(V_i)[G : H]$$

(recall that $[G : H]$ is the index of G in H).

1'.* This is formally a stronger statement than Problem 1, but it may be easier to prove because it does not involve additional information irrelevant for the problem. Let G be a finite group, H a subgroup of G and V an irreducible representation of G over an arbitrary field. Let W be a subspace of V which is H-invariant. Prove that

$$\dim(V) \leq \dim(W)[G : H].$$

2. Let p be a prime and $G = \mathrm{Heis}(\mathbb{Z}_p)$, the Heisenberg group over $\mathbb{Z}_p$ defined in HW#7.2

(a) Determine the number of conjugacy classes of G and their sizes. As in HW#8.6, you can work directly with matrices or with their expressions in terms of the generators x, y, z introduced in HW#7.2.

(b) Let $\omega \neq 1$ be a p^{th} root of unity, that is, $\omega = e^{\frac{2\pi k i}{p}}$ with $1 \leq k \leq p-1$. Let V be a p-dimensional complex vector space with basis $e_{[0]}, e_{[1]}, \ldots, e_{[p-1]}$ where we think of indices as elements of $\mathbb{Z}_p$. Prove that there exists a representation (ρ_ω, V) of G such that

- $\rho_\omega(z)e_{[k]} = \omega e_{[k]}$ for each k (that is, $\rho_\omega(z)$ is just the scalar multiplication by ω),
- $\rho_\omega(y)e_{[k]} = e_{[k+1]}$ for each k (that is, $\rho_\omega(y)$ cyclically permutes the basis vectors) and finally
- $\rho_\omega(x)e_{[k]} = \omega^k e_{[k]}$ for each k.

(c) Prove that every representation in (b) is irreducible (do not do this directly from definition) and every irreducible complex representation of G is either one-dimensional or equivalent to (ρ_ω, V) for some ω.

3. Let (ρ, V) be a representation of a group G. Recall that the dual representation (ρ^*, V^*) is defined by $\rho^*(g)(f) = f \circ \rho(g)^{-1}$ for all $f \in V^*$. Prove parts (1) and (2) of Claim 18.1 from class:

(1) (ρ^*, V^*) is indeed a representation

(2) If β is any basis of V and β^* is the dual basis of V^*, then $[\rho^*(g)]_{\beta^*} = ([\rho(g)]_\beta^{-1})^T$

Note: Part (2) has almost nothing to do with representation theory. Recall from Lecture 9 that given an operator $A \in \text{End}(V)$, its adjoint $A^* \in \text{End}(V^*)$ is defined by $A^*(f) = f \circ A$ (recall that this notion of adjoint is related to but slightly different from adjoints in complex inner product spaces). What you need to prove is Claim 9.2 from class which asserts that $[A^*]_{\beta^*} = ([A]_\beta)^T$ (make sure to explain how (2) follows from this).

4.* Let $G = S_n$ for some $n \geq 2$ and χ an irreducible complex character of G. Prove that χ is real-valued, that is, $\chi(g) \in \mathbb{R}$ for all $g \in G$.

5. Give an example of two representations V and W of the same group which are not equivalent, but have the same character. Recall that by Corollary 19.2 from class this cannot happen if G is finite and representations are complex.

Hint for 1: Consider the special case $H = \{e\}$, the trivial subgroup. In this case each V_i is 1-dimensional, and the problem asserts that $\dim(V) \leq [G : H] = |G|$. We know that the latter is true by HW#8.7 (since irreducible representations are cyclic). This suggests that a suitable generalization of the argument from HW#8.7 may work here.

Hint for 1': If W is any nonzero subspace of V and $Z = \sum_{g \in G} \rho(g)W$, arguing as in HW#8.7, we deduce that Z is G-invariant and hence (since V is irreducible) $Z = V$. However, this only shows that

$$\dim(V) \leq |G| \cdot \dim W.$$

Now argue that if W is already H-invariant, one can get a G-invariant subspace by a similar formula, but taking the sum not over all $g \in G$ but some subset of G (which is small enough to yield the given inequality).

Hint for 4: Use our discussion in Lecture 18 to find a general condition on a finite group G which guarantees that all of its complex characters are real-valued and then show that this condition holds for S_n.