Homework #8. Due Thursday, October 26th

Reading:

1. For this homework assignment: Ben Webster's notes (Lectures 18,19) and class notes (Lectures 16 and 17).

2. Next week we will prove orthogonality relations for characters, that is, parts (1) and (2) of Theorem 17.1. We will roughly follow Ben Webster's Lecture 21. The same material appears in Steinberg's book (4.2, 4.3, 4.4), but in a rather different order.

Problems:

1. Let G be a group, N a normal subgroup of G and let $\pi : G \to G/N$ be the natural projection.

(a) Given a representation $\rho : G/N \to GL(V)$ of G/N, define the representation $\widetilde{\rho} : G \to GL(V)$ of G by

$$\widetilde{\rho}(g) = \rho \circ \pi(g) = \rho(gN). \qquad (***)$$

Prove that $\widetilde{\rho}$ is irreducible $\iff$ ρ is irreducible. Also prove that two representations ρ_1 and ρ_2 of G/N are equivalent $\iff$ the corresponding representations $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ of G are equivalent.

(b) Now fix a field F. Let

- $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible representations of G over F;
- $\mathrm{Irr}(G/N)$ the set of equivalence classes of irreducible representations of G/N over F;
- $\mathrm{Irr}(G, N)$ the set of all $[\rho] \in \mathrm{Irr}(G)$ such that $N \subseteq \mathrm{Ker}\rho$.

(here $[\rho]$ is the equivalence class of the representation ρ). Define the map $\Phi : \mathrm{Irr}(G/N) \to \mathrm{Irr}(G)$ by

$$\Phi([\rho]) = [\widetilde{\rho}]$$

(where $\widetilde{\rho}$ is defined by (***)). Explain why Φ is well defined and injective (this follows immediately from (a)) and then prove that $\mathrm{Im}(\Phi) = \mathrm{Irr}(G, N)$.

Note that we implicitly considered the map Φ in Lecture 14 when we proved that for any group G there is a natural bijection between 1-dimensional representations of G and 1-dimensional representations of its abelianization G^{ab}.

2. Let (ρ_1, V_1) and (ρ_2, V_2) be representations of a group G over the same field. Let $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$ and $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$ be their direct sum and tensor product, respectively. Prove that

(i) $\chi_{\rho_1 \oplus \rho_2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$ for all $g \in G$

(ii) $\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g) \cdot \chi_{\rho_2}(g)$ for all $g \in G$

3. In Lecture 17 we stated what should be the character of the unique (up to equivalence) 2-dimensional irreducible complex representation of S_4 based on our knowledge of the rest of the character table. Now prove this claim using Theorem 17.1. Include all the relevant calculations and try to make your argument as efficient as possible.

4. Compute the character table for a cyclic group of order 3.

5. Compute the character table for the alternating group A_4 (with detailed justification) and explicitly construct its irreducible complex representations. First prove that $[A_4, A_4] = V_4$, the Klein 4-group.

6. Let G be the group of all matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{Z}_5$ and $a \neq 0$.

(a) Prove that G has a presentation $\langle x, y \mid x^4 = y^5 = e, xyx^{-1} = y^2 \rangle$ where $x = \operatorname{diag}(2,1) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $y = E_{12}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (in the notation of HW#7.2).

(b) Prove that G has 5 conjugacy class with sizes $1, 4, 5, 5, 5$. You can use either the original definition or the presentation from part (a).

(c) Now compute the character table of G (with detailed justification).

7. Let G be a finite group and (ρ, V) a cyclic representation of G over an arbitrary field. Prove that $\dim(V) \leq |G|$.