**Homework #6. Due Thursday, October 12th**

**Reading:**

1. For this homework assignment: Ben Webster's notes: lectures 16,17, class notes (Lectures 12 and 13) and Steinberg 3.2, 4.1

2. Next week we will talk about representations of abelian groups and one-dimensional representations (Ben Webster's Lecture 20 and Steinberg, second half of 4.1) and start talking about characters (Ben Webster's Lecture 18 and Steinberg 4.3).

**Problems:**

**1.** The goal of this problem is to prove the general case of Maschke's Theorem: *If $G$ is a finite group and $F$ is any field with* $\mathrm{char}(F) \nmid |G|$, *then any representation of $G$ over $F$ is completely reducible.*

The key part of the proof is the following lemma.

**Lemma:** *Let $G$ and $F$ be as above, $(\rho, V)$ a representation of $G$ over $F$ and $W$ a $G$-invariant subspace. Then there exists a $G$-invariant subspace $U$ such that $V = W \oplus U$.*

Maschke's theorem follows immediately by repeated applications of this lemma (or by induction on $\dim V$).

To prove the lemma, consider the following maps. Choose any subspace $Z$ of $V$ such that $V = W \oplus Z$ and let $P : V \to W$ be the projection onto $W$ along $Z$, that is, $P$ is the unique linear map such that $P(w) = w$ for all $w \in W$ and $P(Z) = 0$. Now define $Q : V \to V$ by

$$Q(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)^{-1} P \rho(g)(v).$$

Prove that

(i) $Q(w) = w$ for all $w \in W$ and $\mathrm{Im}(Q) = W$. Deduce that $Q^2 = Q$.

(ii) $\mathrm{Ker}(Q)$ is $G$-invariant

Deduce from (i) that $V = W \oplus \mathrm{Ker}(Q)$ and therefore $U = \mathrm{Ker}(Q)$ has the desired property.

**2.** Given a vector space $V$, let $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ (we previously denoted this set by $\mathcal{L}(V)$). Elements of $\mathrm{End}(V)$ are called *endomorphisms of $V$*. If $X$

is an algebraic structure (e.g. group, ring, vector space), an endomorphism of $X$ is a homomorphism from $X$ to itself.

(a) Prove that $\mathrm{End}(V)$ is a ring with 1 (where addition is the usual pointwise addition of maps and multiplication is given by composition). Clearly state where you use the fact that elements of $\mathrm{End}(V)$ are linear maps.

Now suppose that $(\rho, V)$ is a representation of some group $G$. Let $\mathrm{End}_\rho(V)$ be the set of those elements of $\mathrm{End}(V)$ which are homomorphisms of representations (from $(\rho, V)$ to $(\rho, V)$). Prove that

(b) $\mathrm{End}_\rho(V)$ is a subring of $\mathrm{End}(V)$ which contains 1 and also that $\mathrm{End}_\rho(V)$ is a vector subspace of $\mathrm{End}(V)$.

(c) If $g \in G$ is a central element, then $\rho(g) \in \mathrm{End}_\rho(V)$.

Note that the claim from the proof of Corollary 13.5 (which was left as an exercise) is an immediate consequence of (b) and (c). Also note that isomorphisms of representations from $(\rho, V)$ to $(\rho, V)$ are precisely invertible elements of $\mathrm{End}_\rho(V)$. Thus Schur's Lemma implies that if $(\rho, V)$ is irreducible, then every nonzero element of $\mathrm{End}_\rho(V)$ is invertible; in other words, $\mathrm{End}_\rho(V)$ is a division ring (axioms of division rings are the same as axioms of fields except that they are not required to be commutative).

**3.** At the beginning of Lecture 14 we will show that if $G$ is an abelian group, then any irreducible representation of $G$ over an algebraically closed field is one-dimensional (this is a straightforward consequence of Corollary 13.5)

(a) Let $n > 2$ be an integer. Construct an irreducible representation of $\mathbb{Z}_n$ over $\mathbb{R}$ (reals) which is not one-dimensional.

(b) Prove that any irreducible representation of $\mathbb{Z}_2$ over any field is one-dimensional.

(c) (bonus) Describe (with proof) all finite abelian groups $G$ such that any irreducible representation of $G$ over any field is one-dimensional.

**4.** Let $(\alpha, V)$ and $(\beta, W)$ be representations of the same group over the same field. The *tensor product* of these representations is the representation $(\rho, V \otimes W)$ where $\rho(g) = \alpha(g) \otimes \beta(g)$ for each $g \in G$ (in the notations from Problem 2 in HW#5). In other words, $\rho(g) \in GL(V \otimes W)$ is the unique linear map such that $\rho(g)(v \otimes w) = (\alpha(g)v) \otimes (\beta(g)w)$ for all $v \in V$ and $w \in W$.

(a) Prove that $(\rho, V \otimes W)$ is indeed a representation, that is, $\rho : G \to GL(V \otimes W)$ is a homomorphism.

(b) Prove that if $(\alpha, V)$ is NOT irreducible and $W \neq 0$, then $(\rho, V \otimes W)$ is not irreducible either.

(c) Now prove that if $(\alpha, V)$ is irreducible and $\dim(W) = 1$, then $(\rho, V \otimes W)$ is irreducible.