Homework #5. Due Friday, October 6th

Reading:

1. For this homework assignment: Ben Webster's notes: lectures 11-15, class notes (Lectures 9-12) and Steinberg 3.1

2. Next Thursday we will prove Maschke's theorem (Ben Webster's Lecture 17 and Steinberg 3.2) and start talking about Schur's lemma (Ben Webster's Lecture 16 and Steinberg 4.1, up to Corollary 4.1.8)

Problems:

For problems (or their parts) marked with a *, a hint is given later in the assignment. Do not to look at the hint(s) until you seriously tried to solve the problem without it.

1. Let U, V and W be vector spaces over the same field F. Construct a natural isomorphism of vector spaces $\phi : (U \oplus V) \otimes W \to (U \otimes W) \oplus (V \otimes W)$ and prove your ϕ is indeed an isomorphism.

2. Let V_1, V_2, W_1 and W_2 be vector spaces over the same field F, and let $\phi : V_1 \to V_2$ and $\psi : W_1 \times W_2$ be linear maps. Prove that there exists a unique linear map $\phi \otimes \psi : V_1 \otimes W_1 \to V_2 \otimes W_2$ such that $(\phi \otimes \psi)(v \otimes w) = \phi(v) \otimes \psi(w)$ for all $v \in V_1$ and $w \in W_1$ (here $\phi \otimes \psi$ is just the notation for the map being defined).

3. Let V and W be vector spaces over the same field F, and let $\phi : V \to V$ and $\psi : W \to W$ be linear maps.

(a) Prove that $\mathrm{Tr}(\phi \otimes \psi) = \mathrm{Tr}(\phi)\mathrm{Tr}(\psi)$

(b)* Assume that ϕ and ψ are both diagonalizable. Prove that $\phi \otimes \psi$ is also diagonalizable and express the eigenvalues of $\phi \otimes \psi$ in terms of the eigenvalues of ϕ and ψ.

4. Let $\rho : S_3 \to GL(\mathbb{C}^3)$ be the representation of S_3 introduced in HW#1.7, and let $W = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\}$. Recall that W is S_3-invariant, and let $\rho_W : S_3 \to GL(W)$ be the corresponding subrepresentation. Find a basis β of W such that the matrix $[\rho_W(g)]_\beta$ has integer entries for all $g \in S_3$ and compute those matrices explicitly (for each $g \in S_3$).

5. Let G be a cyclic group and (ρ, V) an irreducible complex representation of G. Prove that $\dim(V) = 1$.

6. The goal of this problem is to establish the equivalence of the external and internal direct sums of representations (the result of this exercise was implicitly used in Lecture 12).

External direct sum. As in class, given two representations (ρ_1, V_1) and (ρ_2, V_2) of the same group G over the same field, define their external direct sum to be the representation (ρ, V) where $V = V_1 \oplus V_2$ and $\rho : G \to GL(V)$ is given by $\rho(g)((v_1, v_2)) = (\rho_1(g)(v_1), \rho_2(g)(v_2))$.

Internal direct sum. Let (ρ, V) be a representation of a group G, and let V_1 and V_2 be subrepresentations of V (that is, G-invariant subspaces) such that $V = V_1 \oplus V_2$ (as vector spaces). In this case we say that V is an internal direct sum of V_1 and V_2 (as a representations of G).

Prove that the external and internal direct sums are equivalent as representations of G in the following sense. Let (ρ, V) be a representations of G, and let V_1 and V_2 be subrepresentations of V such that $V = V_1 \oplus V_2$. Prove that (ρ, V) is equivalent to the (external) direct sum of the representations (ρ_1, V_1) and (ρ_2, V_2) where $\rho_i(g) \in GL(V_i)$ is simply the restriction of $\rho(g)$ to V_i.

Just in the case of vector spaces, we will not explicitly distinguish between external and internal direct sums in the future identifying them via the above equivalence.

Hint for 3(b): Think of diagonalizability in terms of eigenvectors.