Homework #4. Due Thursday, September 21st

Reading:

1. For this homework assignment: Friedberg-Insel-Spence 6.4, 6.5, Ben Webster's notes: lectures 8,9 + class notes (Lectures 7,8)

2. Next week we will talk about dual spaces (Friedberg-Insel-Spence 2.6 + Ben Webster's Lecture 14) and start talking about tensor products (Ben Webster's Lectures 12,13).

Problems:

For problems (or their parts) marked with a *, a hint is given later in the assignment. Do not to look at the hint(s) until you seriously tried to solve the problem without it.

Note: Problems 1 and 2 establish some fundamental facts about unitary operators that we will continuously use when talking and representations.

0. Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let $A \in \mathcal{L}(V)$ be a normal operator. Prove that A is unitary if and only if all eigenvalues of A have absolute value 1.

1. Let V be an inner product space over $\mathbb{C}$ and $A \in GL(V)$, that is, $A \in \mathcal{L}(V)$ is invertible.

(a) Prove that A is unitary if and only if $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all $x, y \in V$.

(b)* Now use (a) to prove that A is unitary if and only if $\|Ax\| = \|x\|$ for all $x \in V$.

2. Let V be an inner product space over $\mathbb{C}$, let $A \in \mathcal{L}(V)$ be unitary, and let $W \subseteq V$ be a *finite-dimensional* subspace of V which is A-invariant (that is, $A(W) \subseteq W$).

(a) Prove that if $A(W) = W$.

(b) Use (a) to prove that $W^{\perp}$ is also A-invariant.

3. Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find a unitary matrix U such that $U^{-1}AU$ is diagonal (see the online version of Lecture 8 for a brief discussion of the algorithm for finding U). Try to determine the eigenvalues of A without computing the characteristic polynomial.

4. Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let H and G be Hermitian forms on V.

(a)* Assume that G is positive-definite. Prove that there exists a basis β of V such that $[H]_\beta$ and $[G]_\beta$ are both diagonal (equivalently, if $A, B \in Mat_n(\mathbb{C})$ are Hermitian matrices and A is positive definite, there exists $P \in GL_n(\mathbb{C})$ such that P^*AP and P^*BP are both diagonal).

(b) (bonus) Now give an explicit example showing that if neither G nor H is positive-definite, the conclusion of (a) may fail.

5.

(a)* Let $A \in Mat_n(\mathbb{C})$ be arbitrary. Prove that there exists a unitary matrix U such that $U^{-1}AU = U^*AU$ is upper-triangular.

(b)* Use (a) to give a short proof of Theorem 8.5 from class in the case when A is Hermitian or unitary (without referring to Theorem 7.2)

Hint for 1(b): One direction is straightforward. For the other direction use the result of an earlier homework problem.

Hint for 4: The result follows from one of the theorems from class with almost no additional computations (but it may take some work to figure out which result to use).

Hint for 5(a): Reformulate the result in terms of operators and imitate the proof of Theorem 7.2 (there are many steps in that proof that are irrelevant for this problem).

Hint for 5(b): If A is Hermitian or unitary, what can you say about $U^{-1}AU$ for unitary U? Once you answer this question, show that if A is Hermitian or unitary and U is unitary such that $U^{-1}AU$ is upper-triangular, then $U^{-1}AU$ must actually be diagonal.