Homework #2. Due Thursday, September 14th

Reading:

1. For this homework assignment: Friedberg-Insel-Spence 6.1, 6.3, Ben Webster's notes: lectures 5,7,8 + class notes (Lectures 5,6)

2. For the next week's classes: Friedberg-Insel-Spence 6.4, 6.5, Ben Webster's notes: lectures 8,9

Problems:

0. Give a detailed proof of the fact that a real bilinear form with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (with respect to some basis) has signature $(1,1)$. (Many of you stated this in HW#2 without any explanation).

1. Let V be a finite-dimensional vector space over a field F of characteristic 2 and H a symmetric (=skew-symmetric since $\mathrm{char}F = 2$) bilinear form on V. Prove that there exist subspaces V_1 and V_2 of V such that

(a) $V = V_1 \oplus V_2$ and $V_1 \perp V_2$ (that is, $H(v,w) = 0$ for all $v \in V_1$ and $w \in V_2$).

(b) $H_{|V_1}$ is diagonalizable (that is, $[H_{|V_1}]_{\beta_1}$ is diagonal for some basis β_1 of V_1)

(c) $H_{|V_2}$ is alternating and non-degenerate (such a form is called symplectic).

Hint: Combine the proofs of Theorems 3.4 and 5.1 from class.

2. Let H be a bilinear form on a vector space V.

(a) Assume that V is finite-dimensional. Prove that H is left-nondegenerate if and only if H is right-nondegenerate.

(b) (bonus) Construct an example of an infinite-dimensional vector space V and a bilinear form H on V which is left-nondenerate but not right-nondegenerate.

3. Let V be an inner product space.

(a) Prove the parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in V$.

(b) Show that $\langle x, y \rangle$ can be expressed as a linear combination of squares of norms. In Lecture 6 we discussed how to do this for the real inner product spaces.

4. Let V be a finite-dimensional complex inner product space and $A \in \mathcal{L}(V)$. Prove that $\mathrm{Im}(A^*) = \mathrm{Ker}(A)^\perp$ (where the orthogonal complement is with respect to the inner product on V).

5. Let V be an inner product space where $\dim V$ is finite or countable, β an orthonormal basis of V and $A \in \mathcal{L}(V)$.

(a) Prove that if $A^* \in \mathcal{L}(V)$ is any operator such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $x, y \in V$, then $[A^*]_\beta = [A]_\beta^*$ (where $[A]_\beta^*$ is the conjugate transpose of A). In particular, this shows that the adjoint operator is unique (if exists).

(b) As we proved in class, the adjoint A^* always exists if $\dim V$ is finite. Now use (a) and a result from earlier homeworks to show that if V is countably-dimensional, then the adjoint A^* may not exist.