Homework #2. Due Thursday, September 7th

Reading:

1. For this homework assignment: Ben Webster's notes: lectures 3,4 + class notes (Lectures 3,4)

2. For the next week's classes: Ben Webster's notes: lectures 5,7,8 (note that there is no lecture 6)

Problems:

For problems (or their parts) marked with a *, a hint is given later in the assignment. Do not to look at the hint(s) until you seriously tried to solve the problem without it.

1. Let V and H be as in Problem 3 of Homework 1.

(a) Prove that H is positive definite directly from definition. You will need some basic facts from real analysis to make the argument rigorous.

(b) Now use the "modified Gram-Schmidt process" (that is, the algorithm from the proof of Theorem 3.4 from class) to find a basis β such that $[H]_\beta$ is the identity matrix.

2.* Let $V = Mat_n(\mathbb{R})$ for some $n \in \mathbb{N}$, and let H be the bilinear form on V given by $H(A, B) = Tr(AB)$. Prove that H is symmetric and compute its signature. It may be a good idea to start with $n = 2$ and $n = 3$.

3. The goal of this problem is to prove the following theorem:

Theorem: *Let F be a finite field with $\mathrm{char}(F) \neq 2$, V a finite-dimensional vector space and H a symmetric bilinear form on V. Then there exists a basis β of V such that $[H]_\beta$ is diagonal and at MOST one entry of $[H]_\beta$ is different from 0 or 1 (in particular, if H is non-degenerate, then there exists a basis β such that $[H]_\beta = diag(1, \ldots, 1, \lambda)$ for some $\lambda \in F$).*

If you do not feel comfortable working with arbitrary finite fields, you can assume that $F = \mathbb{Z}_p$ for some $p > 2$ (this does not substantially simplify the problem).

(a) * Let Q be the set of squares in F, that is, $Q = \{f \in F : f = x^2 \text{ for some } x \in F\}$. Prove that $|Q| = \frac{|F|+1}{2}$.

(b) * Now take any nonzero $a, b \in F$. Use (a) to prove that there exist $x, y \in F$ such that $ax^2 + by^2 = 1$.

(c) Now use (b) to prove the above Theorem. **Hint:** The main case to consider is when $\dim(V) = 2$ and H is non-degenerate. Once you prove the theorem in this case, the general statement follows fairly easily by induction (using the diagonalization theorem, Theorem 3.4). In the case $\dim(V) = 2$ and H is non-degenerate we already know that there is a basis β such that $[H]_\beta$ is diagonal with nonzero diagonal entries. Now starting with that basis, try to imitate the proof of Theorem 3.4, using (b) at some stage.

4.* Let H be a bilinear form on a finite-dimensional vector space V. In class we proved that for any subspace W of V we have $\dim(W) + \dim(W^\perp) \geq \dim(V)$ (Lemma 3.2) where $W^\perp$ is the orthogonal complement of W with respect to H. Prove that if H is non-degenerate, then $\dim(W) + \dim(W^\perp) = \dim(V)$. One way to prove this is to show that the map ϕ from the proof of Lemma 3.2 is surjective.

5. In this problem we discuss linear maps and bilinear forms on vector spaces of (infinite) countable dimension over an arbitrary field F. One example of such a space is F^∞_{fin}, the set of (infinite) sequences of elements of F in which only finitely many elements are nonzero. The set $\{e_1, e_2, \ldots\}$ is a basis of F^∞_{fin} where e_i is the sequence whose i^{th} element is 1 and all other elements are 0.

Now let V be any countably-dimensional vector space over F and $\beta = \{v_1, v_2, \ldots\}$ a basis of V. Any $v \in V$ is a linear combination of finitely many elements of β, so we can write $v = \sum_{i=1}^n \lambda_i v_i$ for some n (if some v_i with $i \leq n$ does not appear in the expansion of v, we simply let $\lambda_i = 0$). Define $[v]_\beta = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots) \in F^\infty_{fin}$.

(a) (practice) Prove that the map $\phi : V \to F^\infty_{fin}$ given by $\phi(v) = [v]_\beta$ is an isomorphism of vector spaces.

Denote by $Mat_\infty(F)$ the set of all matrices with countably many rows and columns whose entries are in F. Given a bilinear form H on V, let $[H]_\beta \in Mat_\infty(F)$ be the matrix whose (i, j)-entry is $H(v_i, v_j)$.

(b) Prove that $H(v, w) = [v]_\beta^T [H]_\beta] [w]_\beta$ for any $v, w \in V$ (here we consider $[v]_\beta$ and $[w]_\beta$ as columns). In particular, explain why the expression on the right-hand side is well defined even though $[H]_\beta$ is an infinite-size matrix.

(c) Prove that the map $\Phi : Bil(V) \to Mat_\infty(F)$ given by $\Phi(H) = [H]_\beta$ is an isomorphism of vector spaces.

Now let $T \in \mathcal{L}(V)$ be a linear map from V to V. Define $[T]_\beta \in Mat_\infty(F)$ to be the matrix whose i^{th} column is $[Tv_i]_\beta$.

(d) Prove that the map $\Psi : \mathcal{L}(V) \to Mat_\infty(F)$ given by $\Psi(T) = [T]_\beta$ is linear and injective, but not surjective, and explicitly describe its image.

Hint for 2. Start by computing the matrix of H with respect to the "standard" basis $\{e_{ij}\}$. This matrix is not diagonal, but if you order the elements of $\{e_{ij}\}$ in the right way, the matrix will be block-diagonal with blocks of size at most 2.

Hint for 3(a). Show that if F is any field with $\text{char}(F) \neq 2$, then for any nonzero $f \in F$ the equation $x^2 = f$ has either 2 or 0 solutions.

Hint for 3(b). Rewrite the equation as $1 - ax^2 = by^2$ and use a counting argument (what you need from (a) is that more than half of all elements of F are squares).

Hint for 4. Let $\{w_1, \ldots, w_m\}$ be a basis of W, and assume that ϕ from the proof of Lemma 3.2 is not surjective. Show that there exist $\lambda_1, \ldots, \lambda_m \in F$, not all zero, such that $\sum_{i=1}^{m} \lambda_i H(w_i, v) = 0$ for all $v \in V$ and deduce that H must be degenerate.