Homework #11. Due Thursday, November 30th

Reading:

1. For this homework assignment: Ben Webster's notes (Lectures 23), class notes (Lectures 24-25) and Steinberg, Chapter 8.

2. Plan for the remaining classes: At this point I do not have a precise plan for Lecture 26. After the Thanksgiving break (Lectures 27-29) we will briefly talk about the representation theory of Lie Groups, concentrating on the four important examples: $\mathrm{SU}(2), \mathrm{SO}(3), \mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SO}(3,1)$.

Problems:

1. Let G be the group from Homework#8.6, and let $H = \langle y \rangle$ (in the notations of #8.6), that is, H is the subgroup of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Let $\omega \neq 1$ be a 5^{th} root of unity. Let $(\rho_\omega, \mathbb{C})$ be the representation of H on $\mathbb{C}$ where y acts as multiplication by ω, and let (ρ'_ω, Y) be the induced representation of G.

(a) Explicitly compute the matrices $\rho'_\omega(x)$ and $\rho'_\omega(y)$ with respect to a natural basis of Y (here x and y are the generators of G defined in #8.6).

(b) Use the irreducibility criterion from Lecture 25 to show that ρ'_ω is irreducible.

(c) Deduce from (b) that if we start with two non-equivalent representations of H, where H is a subgroup of a group G, then the corresponding induced representations of G may be equivalent.

2. Let G be a finite group and H a subgroup of G. Let ρ be a representation of H and let ρ' be the induced representation of G.

(a) Use Lemma 25.1 to show that $\chi_{\rho'}(g) = \frac{1}{|H|} \sum\limits_{x \in T(g)} \chi_\rho(x^{-1}gx)$ where $T(g) = \{x \in G : x^{-1}gx \in H\}$.

(b)* Use (a) to prove the Frobenius Reciprocity Theorem:
Frobenius Reciprocity Theorem: *Let G be a finite group, H a subgroup of G, let ρ be a complex representation of H and π a complex representation of G. Then*

$$\langle \chi_{Ind\uparrow_H^G \rho}, \chi_\pi \rangle = \langle \chi_\rho, \chi_{\mathrm{Res}\downarrow_H^G \pi} \rangle.$$

3. Let G be a group, H a subgroup of finite index and G/H the set of left cosets of H in G. Recall that G has a natural action on $X = G/H$ given by

$$g.(xH) = (gx)H.$$

Let F be a field and (π, FX) the permutation representation of G corresponding to this action. Prove that π is equivalent to the induced representation $\mathrm{Ind}\uparrow_H^G \rho_{triv}$ where $(\rho_{triv}, F))$ is the trivial representation of H. **Note:** You may want to start with the special case $H = \{e\}$, the trivial subgroup. In this special case the problem asserts that

Theorem: *The regular representation of a finite group G is induced from the trivial representation of the trivial subgroup.*

4. Use the Frobenius Reciprocity Theorem and Problem 3 to give another proof of Proposition 21.3 from class which asserts that if V is an I$\mathbb{C}$R of a finite group G, then the multiplicity of V in the regular representation is equal to $\dim(V)$. You are allowed to use Proposition 21.2.

5. The goal of this problem is to justify the conceptual representation of the induced representation sketched at the end of Lecture 25. Recall the setup. Let G be a finite group, H a subgroup of G and (ρ, V) a representation of H over a field F. Let $W = F[G] \otimes V$. Define the representation (π, W) of G by $\pi(g)(a \otimes v) = (ga) \otimes v$.

(a) Prove that π is well-defined and that π is indeed a representation.

(b) Let Y be the subspace of W spanned by all elements of the form $(ah) \otimes v - a \otimes (\rho(h)v)$ with $a \in F[G], h \in H$ and $v \in V$. Prove that Y is $\pi(G)$-invariant. Deduce that there is a well-defined representation $(\overline{\pi}, W/Y)$ of G given by $\overline{\pi}(g)(w + Y) = \pi(g)(w) + Y$.

(c)* Now let $g_1, \ldots, g_k$ be a system of representatives of the left cosets of H in G and let $V' = \bigoplus_{i=1}^{k} g_i \otimes V$. Let $f : V' \to W/Y$ be the map obtained by composing the natural inclusion of V' into W with the natural projection $W \to W/Y$. Prove that f is an isomorphism of representations between the induced representation $(\mathrm{Ind}\uparrow_H^G \rho, V')$ and $(\overline{\pi}, W/Y)$

Hint for 2(b) Use 2(a) to express the left-hand side as an iterated sum. Then change the order of summation and use the fact that the character of π is invariant under conjugation by elements of G.

Hint for 5(c) Technically the more difficult part is to show that f is an isomorphism of vector spaces. First show that f is surjective. For this use the fact that if $\overline{a \otimes v}$ denotes the natural image of $a \otimes v$ in W/Y, then $\overline{ah \otimes v} = \overline{a \otimes \rho(h)v}$, with $h \in H$.

Then show that Y is actually spanned by the smaller set

$$T = \{g_i h \otimes v - g_i \otimes (\rho(h)v) : 1 \leq i \leq k, h \in H \setminus \{1\}, v \in V\}.$$

Argue that

$$\dim(\mathrm{Span}(T)) \leq [G : H](|H| - 1) \dim V = (|G| - [G : H]) \dim V$$

and deduce that $\dim(W/Y) \geq [G : H] \dim V$. Combined with surjectivity of f, this implies that f is an isomorphism (of vector spaces).