Homework #10. Due Thursday, November 16th

Reading:

1. For this homework assignment: Ben Webster's notes (Lectures 21, 22), class notes (Lectures 20-23) and Steinberg, 7.1 and 7.2.

2. Next week we will talk about induced representations (Ben Webster's Lecture 23 and Chapter 8 in Steinberg). For the initial definition of the induced representations I recommend (the beginning of) the wikipedia article

`https://en.wikipedia.org/wiki/Induced_representation`

Problems:

1. Recall that in HW#7.5 we proved that if (ρ, V) is any cyclic representation of a finite group G over an arbitrary field, then $\dim(V) \leq |G|$. Now prove that if (ρ, V) is irreducible, then $\dim(V) \leq |G| - 1$.

2. The goal of this problem is to explicitly decompose the regular representation $(\rho_{reg}, \mathbb{C}[S_3])$ as a direct sum of irreducible representations of S_3. Recall that S_3 has 3 I$\mathbb{C}$R's: two one-dimensional (the trivial representation and the sign representation) and one two-dimensional (the standard representation) and that by Proposition 21.3 each I$\mathbb{C}$R appears in $\mathbb{C}[S_3]$ with multiplicity equal to its dimension.

(a) Let $H = \langle (1, 2, 3) \rangle$ and consider $(\rho_{reg}, \mathbb{C}[S_3])$ as a representation of H. Prove that $\mathbb{C}[S_3] = V_1 \oplus V_2$ where both V_1 and V_2 are H-invariant and equivalent (as H-representations) to the regular representation of H.

(b) In Lecture 21 we explicitly decomposed $\mathbb{C}[H]$, the regular representation of H, into a direct sum of 3 one-dimensional H-representations. Combining this with (a), we get an explicit decomposition $\mathbb{C}[S_3] = \oplus_{i=1}^{6} W_i$ where each W_i is a one-dimensional and H-invariant.

Show that after a suitable renumbering of $W_1, \ldots, W_6$ the following is true: $W_1 \oplus W_2$ and $W_3 \oplus W_4$ are both irreducible subrepresentations of S_3 (these are the copies of the standard representation we are supposed to get by Proposition 21.3), while $W_5 \oplus W_6$ decomposes (in a different way) into a direct sum of the trivial and the sign representations of S_3.

3. (Steinberg, Exercise 7.9). Suppose that G is a finite group of order n with s conjugacy classes. Suppose that one chooses a pair $(g, h) \in G \times G$ uniformly at random. Prove that the probability g and h commute is $\frac{s}{n}$. **Hint:** Apply Burnside's counting lemma to a suitable action of G. (If this is not enough, read the hint in Steinberg).

4. (Steinberg, Exercise 7.5, reformulated). Before doing this problem read Chapter 7 of Steinberg. Let p be a prime, and let G be the set of all functions from $\mathbb{Z}_p$ to $\mathbb{Z}_p$ which have the form $x \mapsto ax + b$ for some $a \in \mathbb{Z}_p^\times$ and $b \in \mathbb{Z}_p$.

(a) Prove that G is a group (with respect to composition) isomorphic to the group of matrices $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z}_p^\times, b \in \mathbb{Z}_p \right\}$. Note that for $p = 5$ this is the group from HW#8.6.

(b) The group G has a natural action on $\mathbb{Z}_p$ (given by $g.x = g(x)$ for all $g \in G$ and $x \in \mathbb{Z}_p$). Prove that this action is 2-transitive (see Steinberg 7.1 for the definition).

(c) By Lemma 21.2(2) the action from (b) yields a homomorphism $\phi : G \to S_p$. Composing ϕ with the standard representation of S_p, we obtain a $(p-1)$-dimensional representation of G. Deduce from (b) (and a suitable result from Steinberg 7.2) that this representation is irreducible.

5. Let C be the cube in $\mathbb{R}^3$ whose vertices have coordinates $(\pm 1, \pm 1, \pm 1)$. Let G be the group of rotations of C, that is rotations in $\mathbb{R}^3$ which preserve the cube (you may assume that G is a group without proof). Let X be the set of 4 main diagonals of C (diagonals connecting the opposite vertices). Note that G naturally acts on X and therefore we have a homomorphism $\pi : G \to Sym(X) \cong S_4$.

(a)* Prove that π is an isomorphism.

(b) Note that G is naturally a subgroup of $\mathrm{GL}_3(\mathbb{R})$ and hence also a subgroup of $\mathrm{GL}_3(\mathbb{C})$, and let $\iota : G \to \mathrm{GL}_3(\mathbb{C})$ be the inclusion map. By (a) we get a representation $\iota \circ \pi^{-1} : S_4 \to \mathrm{GL}_3(\mathbb{C})$. Prove that this representation is equivalent to the tensor product of the standard and sign representations.

Hint for 5(a) First show that G acts transitively on the 8 vertices of C. Then show that the stabilizer of a fixed vertex had order ≥ 3. This implies that $|G| \geq 24 = |S_4|$. Finally, show that π is injective (since $|G| \geq |S_4|$, this would force π to be an isomorphism).