Homework #1. Due Thursday, August 31st

Reading:

1. For this homework assignment: Ben Webster's notes: lectures 1,2 + class notes (Lectures 1,2)

2. For the next week's classes: Ben Webster's notes: lectures 3-5.

Problems:

Some of the problems below deal with concepts that have not been formally introduced in class so far. The definitions of those concepts are given on page 3. We denote by $Mat_n(F)$ the set of all $n \times n$ matrices over F.

1. Let $V = Pol_2(\mathbb{R})$, the vector space of polynomials of degree at most 2 over $\mathbb{R}$. Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, (x-1), (x-1)^2\}$. Both β and γ are bases of V (you do not need to verify this). Let $T : V \to V$ be the differentiation map: $T(f) = f'$.

(a) compute the matrix $[T]_\beta$ directly from definition

(b) compute the matrix $[T]_\gamma$ directly from definition

(c) now compute $[T]_\gamma$ using your answer in (a) and the change of basis formula.

2. In each of the following examples determine if H is a bilinear form on V (make sure to justify your answer):

(a) $V = Mat_n(F)$ for some field F and $n \in \mathbb{N}$ and $H(A, B) = AB$.

(b) $V = Mat_n(F)$ for some field F and $n \in \mathbb{N}$ and $H(A, B) = (AB)_{1,1}$ (the (1,1)-entry of the matrix AB).

(c) $V = F^n$ for some field F and $n \in \mathbb{N}$ and $H((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1 + y_1$.

3. As in problem 1, let $V = Pol_2(\mathbb{R})$, and define $H : V \times V \to \mathbb{R}$ by

$$H(f, g) = \int_0^1 f(x)g(x)dx.$$

Prove that H is a symmetric bilinear form and compute the matrix $[H]_\beta$ (where again $\beta = \{1, x, x^2\}$).

4. Let F be any field, $n \in \mathbb{N}$ and $V = Mat_n(F)$, the vector space of $n \times n$ matrices over F. Let e_{ij} be the matrix whose (i, j)-entry is equal to 1 and all other entries are 0. Then $\beta = \{e_{ij} : 1 \leq i, j \leq n\}$ is a basis of V (you do not need to verify this). Define $H : V \times V \to F$ by

$$H(A, B) = Tr(AB^T)$$

(where B^T is the transpose of B). Prove that H is a symmetric bilinear form and compute the matrix $[H]_\beta$ (you can order β in any way you like). Include all the relevant computations.

5. Let F be a field with $\mathrm{char}(F) \neq 2$, let V be a finite-dimensional vector space over F, and let H be a bilinear form on V. Prove that H can be **uniquely** written as $H = H^+ + H^-$ where H^+ is a symmetric bilinear form on V and H^- is an antisymmetric bilinear form on V.

6. Let F be any field and $n \in \mathbb{N}$.

(a) Let $V = F^n$ (the standard n-dimensional vector space over F). Let $D : V \times V \to F$ be the dot product form. Prove that D is non-degenerate.

(b)* Now V be any n-dimensional vector space over F, β an ordered basis for V and H a bilinear form on V. Prove that H is left non-degenerate if and only if $[H]_\beta$ (the matrix of H with respect to β) is invertible.

Note: (a) is a special case of (b); however, there is a natural way to solve (b) using (a), so it does make sense to prove (a) first.

7. Let F be any field, $n \in \mathbb{N}$, $V = F^n$ and $\{e_1, \ldots, e_n\}$ the standard basis of V. Define $\rho : S_n \to GL(V)$ by $(\rho(g))(e_i) = e_{g(i)}$. As discussed in Lecture 1, the pair (ρ, V) is a representation of S_n.

(a) Let V_0 be the subspace of V consisting of all vectors whose sum of coordinates is equal to 0:

$$V_0 = \{(x_1, \ldots, x_n) \in V : x_1 + \ldots + x_n = 0\}.$$

Prove that V_0 is an S_n-invariant subspace of V, and therefore (ρ, V_0) is also a representation of S_n.

(b)* **BONUS** Now prove that the representation (ρ, V_0) is irreducible, that is, if W is any S_n-invariant subspace of V_0, then $W = 0$ or $W = V_0$.

Definitions

1. *Characteristic of a ring.* Let R be a ring with 1. The *characteristic* of R, denoted $\mathrm{char}(R)$, is the smallest positive integer n such that $\underbrace{1+\ldots+1}_{n \text{ times}}=0$ in R. If no such n exists, we define $\mathrm{char}(R)=0$. For instance, $\mathrm{char}(\mathbb{Z}) = \mathrm{char}(\mathbb{Q}) = \mathrm{char}(\mathbb{R}) = \mathrm{char}(\mathbb{C}) = 0$, while $\mathrm{char}(\mathbb{Z}_n) = n$ (where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the ring of congruence classes mod n). There is a theorem saying that if F is a field, then $\mathrm{char}(F)$ is either 0 or a prime number.

2. Let H be a bilinear form on a vector space V (over any field). Then H is called

(i) *symmetric* if $H(x,y) = H(y,x)$ for all $x, y \in V$;

(ii) *antisymmetric* if $H(x,y) = -H(y,x)$ for all $x, y \in V$;

(iii) *left non-degenerate* if for every nonzero $x \in V$ there exists $y \in V$ with $H(x,y) \neq 0$.

(iv) *right non-degenerate* if for every nonzero $x \in V$ there exists $y \in V$ with $H(y,x) \neq 0$.

Hint for 6(b). Recall the formula $H(v,w) = [v]_\beta^T [H]_\beta [w]_\beta$. Interpret the right-hand side of this formula as a dot product and use 6(a).

Hint 1 for 7(b). Let W be an S_n-invariant subspace of V_0, and assume that $W \neq 0$. Our goal is to show that $W = V_0$. First prove that W must contain a vector of the form $e_i - e_j$ for some $i \neq j$.

Hint 2 for 7(b) (we continue in the setting of hint 1). This is a hint how to prove that W contains $e_i - e_j$ for some $i \neq j$. Given $w \in V$, let $comp(w)$ (the complexity of w) be the number of distinct nonzero coordinates of w. For instance, $comp(e_i) = 1$ and $comp(e_i - e_j) = 2$ for $i \neq j$. Now let $w \in W$ be the nonzero element of the smallest possible complexity. Prove that $comp(w) = 2$ and deduce that w is a nonzero scalar multiple of $e_i - e_j$ for some $i \neq j$.

Hint 3 for 7(b) (we continue in the setting of hint 2). Finally, here is a hint how to prove that $comp(w) = 2$. First show that $comp(w) > 1$ (this easily follows from the fact that $w \in W \subseteq V_0$). Now assume that $comp(w) > 2$ and use w and S_n-invariance of W to cook another nonzero element of W whose complexity is smaller than the complexity of w, thus reaching a contradiction.