Homework #4. Due Thursday, Feb 20th, by 1pm in my mailbox

Reading and plan for the next week:

1. For this homework assignment read 4.8, 5.1 and parts of 5.2
2. Plan for next week: 5.1-5.4. I am not sure about the order, but we will almost definitely start with the sphere-covering bound (5.2) and sphere-packing bound (5.3)

Problems:

1. Problem 4.27. **Hint:** This problem can be solved using the same idea as 4.20(i), (ii), but there is a more conceptual solution involving cosets.

2. Problem 4.47. In addition to what you are asked to do in the book, construct the syndrome look-up table and use it to decode the three words given in the problem.

3. Problem 4.44.

4. Let $r \geq 2$, let $N = 2^r - 1$ and let $Ham(r, 2)$ be the binary Hamming code of length N (see 5.3.1). Recall that we proved in Lecture 7 that $Ham(r, 2)$ has distance 3.

(a) Consider the following $N+1 = 2^r$ elements: $0, e_1, \ldots, e_N$ (where e_i is the i^{th} element of the standard basis). Prove that every coset of $Ham(r, 2)$ in $\mathbb{F}_2^N$ contains exactly one of these elements.

(b) Deduce from (a) that for every $w \in \mathbb{F}_2^N$ there exists $c \in Ham(r, 2)$ such that $d(w, c) \leq 1$. This implies that $Ham(r, 2)$ is a perfect code (a slightly different proof of this fact will be given in class next week).

5. In parts (a) and (b) of this problem assume that $d \geq 2$.

(a) Prove that $A_q(n, d) \leq A_q(n, d-1)$. In other words, fix an alphabet A with $|A| = q$ and prove the following: if there exists an (n, M, d)-code C over A, there also exists an $(n, M, d-1)$-code C' over A. You should give a precise argument; do not try to say this is obvious or something like that.

(b) Now prove that $A_q(n, d) \leq A_q(n-1, d-1)$. **Hint:** one way to do this is to imitate the second solution to HW 1.1 given in Lecture 9. You will likely need (a) to complete the argument.

(c) Now use (b) and induction to prove the singleton bound: $A_q(n, d) \leq q^{n-d+1}$.

6. You are not allowed to use the Plotkin bound in this problem.

(a) Assume that $\frac{2n}{3} < d \leq n$. Prove that $A_2(n, d) = 2$. **Hint:** Use the same idea as in the proof of the equality $A_2(5, 4) = 2$ given in Lecture 9.

(b) Assume now that $d = \frac{2n}{3}$. Prove that $A_2(n, d) \geq 4$.

(c) (bonus) Now prove that if $d = \frac{2n}{3}$, then $A_2(n, d) = 4$.