Introduction to Real Analysis, Fall 2018. Midterm #2.
Due Friday, November 16th, by 1pm (in my mailbox)

Directions: Provide complete arguments (do not skip steps). State clearly and FULLY any result you are referring to. Partial credit for incorrect solutions, containing steps in the right direction, may be given. If you are unable to solve a problem (or a part of a problem), **you may still use its result to solve a later part of the same problem or a later problem in the exam**.

Scoring system: Exam consists of **4** problems, each worth 12 points. If s_1, s_2, s_3, s_4 are your individual scores in decreasing order, your total is $s_1 + s_2 + s_3 + s_4/2$. Thus, the maximal possible total is 42, but the score of 40 counts as 100%.

Rules: You are NOT allowed to discuss midterm problems with anyone else except me. You may ask me any questions about the problems (e.g. if the formulation is unclear), but I may only provide minor hints. You may freely use your class notes, previous homework assignments (including posted solutions) and class textbooks (Pugh and Rudin). The use of other books or any online sources besides the course webpage is prohibited.

Hint: Some proofs from class are very relevant for this test.

1. Let (X, d) be a metric space, and let $\{f_n : X \to \mathbb{R}\}$ be an equicontinuous sequence of functions.

(a) (4 pts) Let $U = \{x \in X : \text{the sequence } \{f_n(x)\} \text{ is bounded }\}$. Prove that U is open in X

(b) (8 pts) Suppose that X has a dense subset E such that $\{f_n\}$ converges pointwise on E. Prove that $\{f_n\}$ converges pointwise on X.

2. In all parts of this problem X is a set, $B = Func(X, [0, 1])$, the set of all functions from X to $[0, 1]$ (the closed interval $[0, 1]$ in $\mathbb{R}$), $\{f_n\}$ is a sequence in B, and $f \in B$.

(a) (5 pts) Suppose that $X = \{x_1, \ldots, x_n\}$ is finite and define a metric d on B by $d(f, g) = \sum_{i=1}^{n} |f(x_i) - g(x_i)|$ (you do not need to prove that d is a metric). Prove that the following are equivalent:

(i) $\{f_n\}$ converges to f in the metric space (B, d)

(ii) $\{f_n\}$ converges to f uniformly on X

(iii) $\{f_n\}$ converges to f pointwise on X

(b*) (3 pts) Now suppose that X is countably infinite. Define a metric d on B such that conditions (i) and (iii) from (a) are equivalent, that is, $\{f_n\}$ converges to f in the metric space $(B, d) \iff \{f_n\}$ converges to f pointwise on X. A formula for d and a brief explanation are sufficient.

(c*) (4 pts) Now suppose that X is uncountable. Prove that there is no metric d on B such that (i) and (iii) are equivalent (that is, pointwise convergence on X is not equivalent to convergence in any metric on B).

3. The ultimate goal of this problem is to prove a converse of the extreme value theorem.

(a)* (3 pts) Let Y be a metric space and X a subspace of Y which is not closed. Prove that there exists a function $f : X \to \mathbb{R}$ which is continuous and unbounded (f just needs to be defined on X).

(b) (3 pts) Let X be a metric space, let $\delta > 0$, and suppose there exists an infinite sequence $\{x_n\}$ in X such that $d(x_i, x_j) \geq 3\delta$ for all $i \neq j$. Define the function $f : X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} i \cdot (\delta - d(x, x_i)) & \text{if } d(x, x_i) < \delta \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Prove that f is well defined (that is, for each x there is at most one i such that $d(x, x_i) < \delta$) and that f is continuous and unbounded.

(c) (4 pts) Let X be any non-compact metric space. Prove that there exists a continuous unbounded function $f : X \to \mathbb{R}$. **Hint:** consider two cases. In each case the result should follow easily from (a) and (b), respectively.

(d) (2 pts) Now prove that if X is any non-compact metric space, there exists a continuous function $g : X \to \mathbb{R}$ which is bounded but does not have maximum value. **Hint:** You can deduce (d) directly from (c) (you do not need to know anything about the proof of (c); just the result).

4. Let S be a subset of a metric space X. The **boundary** of S, denoted by ∂S, is the set of all $x \in X$ which are contact points for both S and its complement $X \setminus S$.

(a) (3 pts) Prove that ∂S is always closed. **Hint:** This can be done without any computations if you reformulate the definition in a suitable way.

(b) (2 pts) Prove that if S is closed, then $\partial S = S \setminus S^o$ where S^o is the interior of S (that is, the set of interior points of S)

(c) (2 pts) Prove that $\partial\partial S \subseteq \partial S$ for every S (here $\partial\partial S$ means the boundary of the boundary of S)

(d) (2 pts) Give an example where $\partial\partial S \neq \partial S$

(e)* (3 pts) Now prove that $\partial\partial\partial S = \partial\partial S$ for any S.

Hint for 2(b): Think how modify the formula in (a) so that the obtained series will always converge.

Hint for 2(c): By way of contradiction, suppose that such a metric d exists. For each $x \in X$ let $I_x \in B$ be the function given by $I_x(x) = 1$ and $I_x(y) = 0$ if $y \neq x$. First prove that there exists $\varepsilon > 0$ such that the set $\{x \in X : d(I_x, \mathbf{0}) > \varepsilon\}$ is infinite (here $\mathbf{0}$ is the identically zero function) and then construct a sequence $\{f_n\}$ and a function f for which (iii) holds but (i) does not hold.

Hint for 3(a): If you are not sure how to start, first consider the case $X = (0, 1]$, $Y = [0, 1]$.

Hint for 4(e): Use 4(b) twice.