Math 4310, Fall 2018. Solutions to the First Midterm.

2. Let (X, d) be a metric space and Y a subset of X. In this problem you are NOT allowed to use that covering compactness is equivalent to sequential compactness.

(a) (4 pts) Define what it means for Y to be covering compact (if you are talking about open covers, define what it means).
(b) (3 pts) Prove that if Y is finite, then Y is covering compact.
(c) (5 pts) Now assume that the metric d on X is discrete ($d(x, y) = 1$ for all $x \neq y$). Prove that in this case the converse of (b) holds: if Y is covering compact, then Y is finite.

Solution: (a) An open cover of Y is a collection $\{U_\alpha\}$ of open subsets of X such that $Y \subseteq \cup U_\alpha$.

We say that Y is covering compact if for any open cover $\{U_\alpha\}$ of Y there exist finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $Y \subseteq \bigcup\limits_{k=1}^{n} U_{\alpha_k}$.

Remark: Many exam papers seemed to be confusing of an open cover $\{U_\alpha\}$ with the union of its elements $\cup U_\alpha$. The union $\cup U_\alpha$ is just a subset of X. There are many different open covers for which the union $\cup U_\alpha$ will be the same (in fact, in the special case $Y = X$ we will have $\cup U_\alpha = Y = X$ for any open cover). An open cover itself is a COLLECTION of subsets of X; in other words, an open cover is a subset of $\mathcal{P}(X)$, the power set of X.

(b) Since Y is finite, we can list of its elements: $Y = \{y_1, \ldots, y_n\}$. Let $\{U_\alpha\}$ be any open cover of Y. Since $Y \subseteq \cup U_\alpha$, for each $1 \leq k \leq n$ there exists an index α_k such that $y_k \in U_{\alpha_k}$. But then $Y = \bigcup\limits_{k=1}^{n} \{y_k\} \subseteq \bigcup\limits_{k=1}^{n} U_{\alpha_k}$, so by definition Y is covering compact.

(c) We will prove (c) by contrapositive: if Y is infinite, then Y is not covering compact. For each $y \in Y$ let $U_y = N_1(y)$, the open ball of radius 1 around y. Then each U_y is open and $Y \subseteq \cup_{y \in Y} U_y$ (since $y \in U_y$ for each y), so the collection $\{U_y\}_{y \in Y}$ is an open cover of Y (what we said so far is true for any metric space).

Now since the metric on X is discrete, for each $y \in Y$ we have $U_y = \{y\}$. Hence, if we pick any finitely many elements $y_1, \ldots, y_n \in Y$, the union $\bigcup\limits_{k=1}^{n} U_{y_k}$ is finite and cannot contain Y. Thus, $\{U_y\}_{y \in Y}$ is an open cover of Y without a finite subcover, so Y is not covering compact.

3. Let $X = C[-1, 1]$ denote the space of continuous functions on $[-1, 1]$. Recall that the uniform metric d_{unif} on X is defined by

$$d_{unif}(u, v) = \max_{x \in [-1,1]} |u(x) - v(x)| \quad \forall u, v \in C[-1, 1]$$

and the integral metric d_{int} on X is defined by

$$d_{int}(u,v) = \int_{-1}^{1} |u(x) - v(x)|\,dx \quad \forall u, v \in C[-1,1]$$

Define the function $F : X \to \mathbb{R}$ by

$$F(u) = u(0),$$

that is, F maps every element element of X (which itself is a function from $[-1,1]$ to $\mathbb{R}$) to its value at 0. For instance, if $u(x) = 2x+1$, then $F(u) = 2 \cdot 0 + 1 = 1$

(a) (6 pts) Prove that if we consider X as a metric space with uniform metric and $\mathbb{R}$ with the standard metric, then F is continuous.

(b) (6 pts) Prove that if we consider X as a metric space with integral metric and $\mathbb{R}$ with the standard metric, then F is not continuous at $\mathbf{0}$ (where $\mathbf{0}$ is the function which is identically zero).

Solution: (a) Take any $u, v \in X$. Then

$$d_{\mathbb{R}}(F(u), F(v)) = |F(u)-F(v)| = |u(0)-v(0)| \le \max_{x\in[-1,1]} |u(x)-v(x)| = d_X(u,v).$$

Hence F is continuous by Problem 6(a) in HW#3.

(b) For each $n \in \mathbb{N}$ let $u_n : [-1,1] \to \mathbb{R}$ be the piecewise linear function given by

$$u_n(x) = \begin{cases} 0 & \text{if } |x| > \frac{1}{n} \\ 1 - n|x| & \text{if } |x| \le \frac{1}{n} \end{cases}$$

By construction u_n is continuous. The region under the graph of u_n is a triangle with height 1 and width $\frac{2}{n}$, so

$$d_{int}(u_n, \mathbf{0}) = \int_{-1}^{1} |u(x) - 0| = \int_{-1}^{1} u(x) = \frac{1}{n}.$$

Thus, $d_{int}(u_n, \mathbf{0}) \to 0$ as $n \to \infty$, so $\{u_n\}$ converges to $\mathbf{0}$ in X.

On the other hand, for all n we have $F(u_n) = u_n(0) = 1$, so $F(u_n)$ does not converge to $F(\mathbf{0})$. Hence F is not continuous at $\mathbf{0}$.

4.

(a) (3 pts) Give an example of a metric space X and a sequence $\{x_n\}$ in X which is Cauchy but not convergent (no proof is needed)

(b) (3 pts) Let S be a subset of a metric space X. Then S is compact if and only if S is closed (in X) and bounded. Prove (if this is true in general) or give a counterexample.

(c) (3 pts) Let (X,d) be a metric space. Suppose that there exists $x_0 \in X$ such that for any $R > 0$, the open ball $N_R(x_0)$ contains only finitely many points. Is it true that X is countable? Prove or give a counterexample.

(d) (3 pts) Let S be a subset of a metric space X. If S is countable, is it always true that $\overline{S}$ (the closure of S in X) is also countable? Prove or give a counterexample.

Solution: (a) For instance, we can take $X = \mathbb{Q}$ (with the standard metric) and let $\{x_n\}$ be any sequence of rationals converging to $\sqrt{2}$. Another simple example is $X = (0, 1)$ and $x_n = \frac{1}{n}$.

(b) This is false. For instance, if we take $X = S = (0, 1)$, then S closed in X (any metric space is a closed subset of itself) and S is bounded, but S is not compact (since it is not closed in $\mathbb{R}$).

Another example: let S be any infinite set with discrete metric. Then S is bounded and S is closed inside any metric space containing it (since any convergent sequence in S is eventually constant by HW#3.5), but S is not compact by Problem 2(c) on this exam.

(c) This is true. Take any $x \in X$. Then $d(x, x_0)$ is a real number, so there exists $n \in \mathbb{N}$ such that $d(x, x_0) < n$ and hence $x \in N_n(x_0)$. Thus, $X = \cup_{n=1}^{\infty} N_n(x_0)$. By assumption each $N_n(x_0)$ is finite (hence countable), so $X = \cup_{n=1}^{\infty} N_n(x_0)$ is a countable union of countable sets and hence X itself is countable by Lecture 3.

(d) This is false. For instance, let $S = \mathbb{Q}$ and $X = \mathbb{R}$. Then S is countable, but $\overline{S} = X$ is uncountable.