Math 4310, Fall 2015. First Midterm.
Thursday, October 8th, 2-3:20 pm

Directions: No books, notes, calculators, laptops, PDAs, cellphones, web appliances, or similar aids are allowed. All work must be your individual efforts. Write your answers and all accompanying work neatly on these pages.

- Show all your work and justify all statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

Scoring system: Exam consists of **4** problems, each of which is worth **12** points. If $k_1 \geq k_2 \geq k_3 \geq k_4$ are your scores on these problems in decreasing order, then your total score will be given by the expression

$$k_1 + k_2 + k_3 + k_4/2$$

Thus, the maximal total is 42, but the score of 40 will be considered 100% (thus, to get 90% it suffices to solve 3 problems correctly).

1.

(a) (3 pts) Give the definition of uniform continuity: Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called uniformly continuous if ...

(b) (9 pts) Let (X, d) be a metric space, fix $a \in X$ and define $f : X \to \mathbb{R}$ by

$$f(x) = d(x, a).$$

Prove that f is uniformly continuous (as usual, the metric on $\mathbb{R}$ is standard).

2. Let X be a metric space.

(a) (3 pts) Let Y be a subset of X. Define what it means for Y to be compact (if you are talking about open covers, define what it means).

(b) (9 pts) Let $\{x_n\}$ be a convergent sequence in X and $x = \lim_{n\to\infty} x_n$. Prove that the set $\{x_n\}_{n\in\mathbb{N}} \cup \{x\}$ is compact. You are not allowed to use that sequential compactness implies compactness. **Hint:** There was a similar homework problem.

3. Let X and Y be metric spaces, $f : X \to Y$ a continuous function and S a subset of X.

(a) (8 pts) Prove that

$$f(\overline{S}) \subseteq \overline{f(S)}$$

(where $\overline{S}$ is the closure of S in X and $\overline{f(S)}$ is the closure of $f(S)$ in Y). **Hint:** Do one of the following:

(i) use sequential characterizations of closures and continuity

(ii) use characterizations of closures and continuity in terms of closed sets or

(iii) assume that there exists $y \in f(\overline{S}) \setminus \overline{f(S)}$ and reach a contradiction using the ε-δ definition of continuity.

(b) (4 pts) Give an example where

$$f(\overline{S}) \neq \overline{f(S)}$$

and briefly explain why your example has the required property.

4. For each of the following statements determine whether it is true (in all cases) or false (in at least one case). If the statement is true, briefly explain why; if not, give a counterexample (and prove that it is a counterexample). An answer (correct or incorrect) without any explanation will not receive any credit.

(a) (3 pts) Every subset of $\mathbb{R}$ (with usual metric) is open or closed.

(b) (3 pts) If X is a metric space and S is a finite subset of X, then S is closed.

(c) (3 pts) Let $\{C_n\}_{n=1}^{\infty}$ be a countable collection of closed subsets of $\mathbb{R}$, and let Irr be the set of all irrational numbers in $\mathbb{R}$. Then

$$\cap_{n=1}^{\infty} C_n \neq Irr.$$

(d) (3 pts) Let $\{U_n\}_{n=1}^{\infty}$ be a countable collection of open subsets of $\mathbb{R}$. Then

$$\cap_{n=1}^{\infty} U_n \neq Irr.$$