Math 5310. Intro to Real Analysis. Fall 2014. First Midterm.

Problem 1: (12 pts) Let A and B be non-empty subsets of $\mathbb{R}$ bounded from above. For each of the following statements determine whether it is true (in general) or false (in at least one case). If the statement is true, prove it; if false, give a counterexample.

(i) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$

(ii) $\sup(A \cap B) = \min\{\sup(A), \sup(B)\}$ provided $A \cap B \neq \emptyset$

(iii) $\sup(A + B) = \sup(A) + \sup(B)$ where $A + B = \{a + b : a \in A, b \in B\}$.

Problem 2: In this problem you are not allowed to refer to the results of homework problems. Let Y be an infinite subset of $[0, 1]$ such that the intersection $Y \cap (\delta, 1]$ is finite for every $\delta > 0$. (Note: $\{1, 1/2, 1/3, \ldots\}$ is an example of such subset).

(a) (4 pts) Prove that Y is countable.

(b) (8 pts) Prove that Y is compact if and only if $0 \in Y$.

Problem 3:

(a) (4 pts) Let (Y, D) be a metric space. Prove that for any $a, b \in Y$, with $a \neq b$, there exists $\varepsilon > 0$ such that the open balls $N_\varepsilon(a)$ and $N_\varepsilon(b)$ have empty intersection.

(b) (8 pts) Let (X, d) and (Y, D) be metric spaces, let $f, g : X \to Y$ be continuous functions, and let

$$K = \{x \in X : f(x) = g(x)\}.$$

Prove that K is a closed subset of X. **Note:** (a) may or may not be useful depending on your approach.

Problem 4: Let (X, d) be an ultrametric space. Recall that this means that (X, d) is a metric space satisfying a stronger version of the triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \text{ for all } x, y, z \in X.$$

(a) (5 pts) Fix $a \in X$ and $\varepsilon > 0$, and let

$$C_\varepsilon(a) = X \setminus N_\varepsilon(a) = \{x \in X : d(x, a) \geq \varepsilon\}.$$

Prove that $C_\varepsilon(a)$ is open in X.

(b) (4 pts) Use (a) to prove that if $|X| \geq 2$, then X is disconnected. Moreover, deduce that any subset Y of X, with $|Y| \geq 2$, is disconnected.

(c) (3 pts) Let $\mathbb{R}$ denote reals with standard metric. Now use (b) to prove that any continuous function $f : \mathbb{R} \to X$ is constant (that is, there exists $x \in X$ such that $f(t) = x$ for all $t \in \mathbb{R}$).