Math 4310, Fall 2015. Solutions to the first midterm.

1.

(a) (3 pts) Give the definition of uniform continuity: Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called uniformly continuous if ...

(b) (9 pts) Let (X, d) be a metric space, fix $a \in X$ and define $f : X \to \mathbb{R}$ by

$$f(x) = d(x, a).$$

Prove that f is uniformly continuous (as usual, the metric on $\mathbb{R}$ is standard).

Solution: (b) By definition we need to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \text{ implies } |f(x) - f(y)| < \varepsilon \text{ for all } x, y \in X \qquad (***)$$

We will show that (***) holds with $\delta = \varepsilon$.

Indeed, take any $x, y \in X$. Then $d(x, a) \le d(x, y) + d(y, a)$ and $d(y, a) \le d(y, x) + d(x, a) = d(x, y) + d(x, a)$. Hence $d(x, a) - d(y, a) \le d(x, y)$ and $d(y, a) - d(x, a) \le d(x, y)$. Therefore,

$$|f(x)-f(y)| = |d(x, a)-d(y, a)| = \max\{d(x, a)-d(y, a), d(y, a)-d(x, a)\} \le d(x, y).$$

Hence $d(x, y) < \varepsilon$ implies $|f(x) - f(y)| < \varepsilon$, so (***) holds with $\delta = \varepsilon$.

2. Let X be a metric space.

(a) (3 pts) Let Y be a subset of X. Define what it means for Y to be compact (if you are talking about open covers, define what it means).

(b) (9 pts) Let $\{x_n\}$ be a convergent sequence in X and $x = \lim_{n\to\infty} x_n$. Prove that the set $\{x_n\}_{n\in\mathbb{N}} \cup \{x\}$ is compact. You are not allowed to use that sequential compactness implies compactness. **Hint:** There was a similar homework problem.

Solution: (a) An open cover of Y is a collection of open subsets $\{U_\alpha\}$ of X such that $Y \subseteq \cup U_\alpha$. The set Y is called compact if for any open cover $\{U_\alpha\}$ of Y there exist finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $Y \subseteq \bigcup_{k=1}^{n} U_{\alpha_k}$.

(b) Let $Y = \{x_n : n \in \mathbb{N}\} \cup \{x\}$, and let $\{U_\alpha\}$ be an open cover of Y. We know that $x \in U_\beta$ for some β. Since U_β is open in X, there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq U_\beta$. Since $x_n \to x$, there exists $M \in \mathbb{N}$ such that $x_n \in N_\varepsilon(x)$ for all $n \ge M$.

It follows that U_β contains x as well as x_n for all $n \ge M$, that is U_β contains all but finitely many elements of Y (the only elements of Y which possibly lie outside of Y are $x_1, \ldots, x_{M-1}$). For each $1 \le i \le M - 1$ choose α_i such that $x_i \in U_{\alpha_i}$. Then by construction $Y \subseteq U_\beta \cup (\cup_{i=1}^{M-1} U_{\alpha_i})$, so we constructed a finite subcover of $\{U_\alpha\}$. Therefore, Y is compact.

3. Let X and Y be metric spaces, $f : X \to Y$ a continuous function and S a subset of X.

(a) (8 pts) Prove that

$$f(\overline{S}) \subseteq \overline{f(S)}$$

(where $\overline{S}$ is the closure of S in X and $\overline{f(S)}$ is the closure of $f(S)$ in Y). **Hint:** Do one of the following:

(i) use sequential characterizations of closures and continuity

(ii) use characterizations of closures and continuity in terms of closed sets or

(iii) assume that there exists $y \in f(\overline{S}) \setminus \overline{f(S)}$ and reach a contradiction using the ε-δ definition of continuity.

(b) (4 pts) Give an example where

$$f(\overline{S}) \neq \overline{f(S)}$$

and briefly explain why your example has the required property.

Solution: (a) First solution (using hint (i)): Take any $x \in \overline{S}$. By Lemma 6.2 (sequential characterization of closures) there exists a sequence $\{s_n\}$ in S such that $s_n \to x$. Since f is continuous at x, by Theorem 6.3 $f(s_n)$ converges to $f(x)$. Since $f(s_n) \in f(S)$, applying Lemma 6.2 again (this time in the opposite direction) we deduce that $f(x) = \lim_{n\to\infty} f(s_n) \in \overline{f(S)}$.

Thus, we showed that $f(x) \in \overline{f(S)}$ for every $x \in \overline{S}$, whence $f(\overline{S}) \subseteq \overline{f(S)}$.

(b) Second solution (using hint (ii)): We know that $\overline{f(S)}$ is closed (the closure of any set is closed). Since f is continuous, $f^{-1}(\overline{f(S)})$ is closed. Since $f(S) \subseteq \overline{f(S)}$, we have $S \subseteq f^{-1}(\overline{f(S)})$. Since the closure of S is contained in any closed subset containing S (Theorem 5.1(3)), we conclude that $\overline{S} \subseteq f^{-1}(\overline{f(S)})$, which by the definition of preimage means that $f(\overline{S}) \subseteq \overline{f(S)}$.

(c) Third solution (using hint (iii)): Proof by contradiction. Suppose that there exists $y \in f(\overline{S}) \setminus \overline{f(S)}$. Thus, $y = f(x)$ for some $x \in \overline{S}$ and $y \notin \overline{f(S)}$. The latter condition means that there exists $\varepsilon > 0$ such that $N_\varepsilon(y) \cap f(S) = \emptyset$.

Since f is continuous at x, there exists $\delta > 0$ such that $f(N_\delta(x)) \subseteq N_\varepsilon(f(x)) = N_\varepsilon(y)$. Thus, $f(N_\delta(x)) \cap f(S) = \emptyset$. This clearly implies that $N_\delta(x) \cap S = \emptyset$, hence by definition $x \notin \overline{S}$, contrary to our initial hypothesis.

(b) We give two types of examples.

Example 1: Let $X = Y = \mathbb{R}$ with standard metric, let $S = \mathbb{R}$ and $f : X \to Y$ any continuous function with non-closed image (for instance, $f(x) = \frac{1}{1+x^2}$ whose image is $(0,1]$). Then $\overline{S} = S = \mathbb{R}$, so $f(\overline{S}) = f(\mathbb{R})$ while $\overline{f(S)} = \overline{f(\mathbb{R})} \neq f(\mathbb{R})$ since $f(\mathbb{R})$ is not closed.

Example 2: Let $Y = \mathbb{R}$ with standard metric, let X be any non-closed subset of $\mathbb{R}$ (also with standard metric), let $S = X$ and let $f : X \to Y$ by the identity function ($f(x) = x$ for all $x \in X$). Then $\overline{S} = S = X$ (since we are taking the closure in X, not in $\mathbb{R}$), so $f(\overline{S}) = X$ while $\overline{f(S)}$ is the closure of X in $\mathbb{R}$ which is strictly larger than X (since X is not closed in $\mathbb{R}$).

4. For each of the following statements determine whether it is true (in all cases) or false (in at least one case). If the statement is true, briefly explain why; if not, give a counterexample (and prove that it is a counterexample). An answer (correct or incorrect) without any explanation will not receive any credit.

(a) (3 pts) Every subset of $\mathbb{R}$ (with usual metric) is open or closed.

(b) (3 pts) If X is a metric space and S is a finite subset of X, then S is closed.

(c) (3 pts) Let $\{C_n\}_{n=1}^{\infty}$ be a countable collection of closed subsets of $\mathbb{R}$, and let Irr be the set of all irrational numbers in $\mathbb{R}$. Then

$$\cap_{n=1}^{\infty} C_n \neq Irr.$$

(d) (3 pts) Let $\{U_n\}_{n=1}^{\infty}$ be a countable collection of open subsets of $\mathbb{R}$. Then

$$\cap_{n=1}^{\infty} U_n \neq Irr.$$

Solution: (a) False. For instance, any half-open interval $(a, b]$ with $a < b$ is neither open nor closed.

(b) True. First argument: We know that points (subsets with one element) are closed (for instance, since every $x \in X$ is equal to $B_0(x)$, the closed ball of radius 0 around x, and closed balls are closed by Homework#3.) Since every finite set is a union of finitely many points and finite unions of closed sets are closed, it follows that finite sets are closed.

Second argument. Let $S = \{s_1, \ldots, s_n\}$. We will show that every $x \notin S$ is not a contact point of S (whence S is closed). Take any $x \notin S$, let $\delta_k = d(x, s_k)$ for $1 \leq k \leq n$ and $\delta = \min\{\delta_1, \ldots, \delta_n\}$. Since each $s_k \neq x$, each $\delta_k > 0$, whence $\delta > 0$. By definition of δ we have $N_\delta(x) \cap S = \emptyset$, so x is not a contact point of S.

Third argument: If S is finite, then any open cover of S clearly has a finite subcover (for each $s \in S$ we just need to pick one element of the cover which contains s). Thus, by definition S is compact, whence S is closed in X by Theorem 8.4.

(c) True. The set $\cap_{n=1}^{\infty} C_n$ is closed since intersection of any collection of closed sets is closed (the fact that the intersection is countable is not essential here). On the other hand, Irr is not closed since $\mathbb{Q}$ is not open (which, in turn, holds since any non-empty open set must contain a non-empty open interval and any non-empty open interval must contain an irrational number).

(d) False, that is, there exists a countable collection of open subsets $\{U_n\}_{n=1}^{\infty}$ such that $\cap U_n = \text{Irr}$.

We know that $\mathbb{Q}$ is countable, so it can be enumerated: $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Let $U_n = \mathbb{R} \setminus \{q_n\} = (-\infty, q_n) \cup (q_n, +\infty)$. Clearly, U_n is open and $\cap U_n = \mathbb{R} \setminus \cup_{n=1}^{\infty} \{q_n\} = \mathbb{R} \setminus \mathbb{Q} = \mathrm{Irr}$, as desired.