Solutions to the First Midterm.

Problem 1: Let A and B be non-empty subsets of $\mathbb{R}$ bounded from above. For each of the following statements determine whether it is true (in general) or false (in at least one case). If the statement is true, prove it; if false, give a counterexample.

(i) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$
(ii) $\sup(A \cap B) = \min\{\sup(A), \sup(B)\}$ provided $A \cap B \neq \emptyset$
(iii) $\sup(A+B) = \sup(A)+\sup(B)$ where $A+B = \{a+b : a \in A, b \in B\}$.

Solution: (i) True. We will first show that $\sup(A\cup B) \leq \max\{\sup(A), \sup(B)\}$ and then that $\max\{\sup(A), \sup(B)\} \leq \sup(A \cup B)$.

Take any $x \in A \cup B$. Then $x \in A$ or $x \in B$, so $x \leq \sup(A)$ or $x \leq \sup(B)$; in either case, $x \leq \max\{\sup(A), \sup(B)\}$. Thus, $\sup(A \cup B) \leq \max\{\sup(A), \sup(B)\}$.

On the other hand, $\sup(A \cup B)$ is an upper bound for $A \cup B$, hence it is an upper bound for both A and B. By the definition of the least upper bound we have $\sup(A) \leq \sup(A \cup B)$ and $\sup(B) \leq \sup(A \cup B)$, whence $\max\{\sup(A), \sup(B)\} \leq \sup(A \cup B)$.

(ii) False. For instance, let $A = \{0, 1\}$, $B = \{0, 2\}$ (both two element sets). Then $A \cap B = \{0\}$, so $\sup(A \cap B) = 0$, while $\min\{\sup(A), \sup(B)\} = \min\{1, 2\} = 1$.

(iii) True. For any $a \in A$ and $b \in B$ we have $a + b \leq \sup(A) + \sup(B)$, so $\sup(A + B) \leq \sup(A) + \sup(B)$. To prove the reverse inequality, $\sup(A) + \sup(B) \leq \sup(A+B)$, we will show that $\sup(A)+\sup(B)-\varepsilon < \sup(A+B)$ for any $\varepsilon > 0$.

So take $\varepsilon > 0$. Since $\sup(A)$ is the least upper bound for A, there exists $a \in A$ such that $\sup(A) - \frac{\varepsilon}{2} < a$, and similarly there exists $b \in B$ such that $\sup(B)-\frac{\varepsilon}{2} < b$. Adding those two inequalities, we get $\sup(A)+\sup(B)-\varepsilon < a + b \leq \sup(A) + \sup(B)$, as desired.

Problem 2: In this problem you are not allowed to refer to the results of homework problems. Let Y be an infinite subset of $[0, 1]$ such that the intersection $Y \cap (\delta, 1]$ is finite for every $\delta > 0$. (Note: $\{1, 1/2, 1/3, \ldots\}$ is an example of such subset).

(a) (4 pts) Prove that Y is countable.
(b) (8 pts) Prove that Y is compact if and only if $0 \in Y$.

Solution: (a) For each $n \in \mathbb{N}$ let $Y_n = Y \cap (\frac{1}{n}, 1]$; thus by assumption each Y_n is finite. Since $\cup_{n\in\mathbb{N}}(\frac{1}{n}, 1] = (0, 1]$, the set $Y \cap (0, 1] = \cup_{n\in\mathbb{N}} Y_n$ is a countable union of finite sets, hence countable. Finally, Y is equal to either $Y \cap (0, 1]$ or $(Y \cap (0, 1]) \cup \{0\}$, so Y is also countable.

(b) We give two solutions – one using the definition of compactness and one using Heine-Borel theorem.

Solution 1: "$\Rightarrow$" We argue by contradiction. Assume that Y is compact, but $0 \notin Y$. Then $Y = \cup_{n\in\mathbb{N}} Y_n$ where $Y_n = Y \cap (\frac{1}{n}, 1]$. Since Y_n also equals $Y \cap (\frac{1}{n}, 2)$, each Y_n is open in Y, so $\{Y_n\}$ is an open cover of Y. By compactness, it must have a finite subcover $Y_{n_1}, \ldots, Y_{n_k}$. But each Y_{n_i} is finite, so this would force Y to be finite as well, contrary to our hypothesis.

"$\Leftarrow$" Assume that $0 \in Y$, and let $\{U_\alpha\}$ be any open cover of Y. Then there exists β such that $0 \in U_\beta$, and since U_β is open in Y, it must contain $Y \cap [0, \varepsilon)$ for some $\varepsilon > 0$. If we now let $\delta = \varepsilon/2$, we get that $Y = U_\beta \cup (Y \cap (\delta, 1])$. By assumption $Y \cap (\delta, 1]$ is finite. If $y_1, \ldots, y_m$ are the elements of $Y \cap (\delta, 1]$, choose indices $\alpha_1, \ldots, \alpha_m$ such that $y_i \in U_{\alpha_i}$. Then $Y = (\cup_{i=1}^m U_{\alpha_i}) \cup U_\beta$, so we found a finite subcover. Theferore, Y is compact.

Solution 2: Y is always bounded, being a subset of $[0, 1]$, so by Heine-Borel theorem Y is compact $\iff$ it is closed ($\iff$ Y contains all its cluster points). We will show that 0 is always a cluster point of Y, and there are no other cluster points. This clearly implies the assertion of the problem.

(i) Proving that 0 is a cluster point of Y. We need to show that for any $\varepsilon > 0$, the set $N_\varepsilon(0) = (-\varepsilon, \varepsilon)$ contains a point of Y other than 0. Note that $Y = (Y \cap (-\varepsilon, \varepsilon)) \cup (Y \cap (\frac{\varepsilon}{2}, 1])$. Since by assumption Y is infinite while $Y \cap (\frac{\varepsilon}{2}, 1]$ is finite, $Y \cap (-\varepsilon, \varepsilon)$ must be infinite; in particular, it must contain a point other than 0. Thus, 0 is indeed a cluster point of Y.

(ii) Proving that $x \neq 0$ is not a cluster point of Y. Since $Y \subseteq [0, 1]$ and $[0, 1]$ is closed, it is clear that any x outside of $[0, 1]$ is not a cluster point. So, suppose that $x \in (0, 1]$, and let $U = N_{\frac{x}{2}}(x) = (\frac{x}{2}, \frac{3x}{2})$. Then $Y \cap U = Y \cap (\frac{x}{2}, 1]$ is finite by assumption, so in particular $Y \cap U$ contains only finitely many points of Y different from x. If $y_1, \ldots, y_k$ are those points and $\delta = \min\{|x-y_i|\}_{i=1}^k$ (which is positive), then the set $N_\delta(x) = (x-\delta, x+\delta)$ does not contain any points of Y other than x, so x cannot be a cluster point of Y.

Problem 3:

(a) (4 pts) Let (Y, D) be a metric space. Prove that for any $a, b \in Y$, with $a \neq b$, there exists $\varepsilon > 0$ such that the open balls $N_\varepsilon(a)$ and $N_\varepsilon(b)$ have empty intersection.

(b) (8 pts) Let (X, d) and (Y, D) be metric spaces, let $f, g : X \to Y$ be continuous functions, and let

$$K = \{x \in X : f(x) = g(x)\}.$$

Prove that K is a closed subset of X. **Note:** (a) may or may not be useful depending on your approach.

Solution: (a) Let $\varepsilon = \frac{d(a,b)}{2}$. If there exists $x \in N_\varepsilon(a) \cap N_\varepsilon(b)$, then $d(x, a) < \varepsilon$ and $d(x, b) < \varepsilon$, whence $d(x, a) + d(x, b) < 2\varepsilon = d(a, b)$, which contradicts the triangle inequality. Therefore, $N_\varepsilon(a) \cap N_\varepsilon(b) = \emptyset$.

(b) **Solution 1:** (this solution does not use the result of (a)). We will prove that K is closed by showing that if a sequence $\{k_n\}$ in K converges to some $x \in X$, then $x \in K$. This follows immediately from the sequential characterization of continuity.

Indeed, suppose $k_n \to x$, with $k_n \in K$. Since f and g are continuous, we must have $f(k_n) \to f(x)$ and $g(k_n) \to g(x)$. Since $k_n \in K$ for all n, we have $f(k_n) = g(k_n)$ by definition of K. Thus the sequences $\{f(k_n)\}$ and $\{g(k_n)\}$ coincide, hence so do their limits: $f(x) = g(x)$. Therefore, $x \in K$.

Solution 2: (this solution does use (a)). We will argue that the complement $X \setminus K$ is open. Take any $x \in X \setminus K$. Then $f(x) \neq g(x)$, so by (a) there exists $\varepsilon > 0$ such that $N^Y_\varepsilon(f(x)) \cap N^Y_\varepsilon(g(x)) = \emptyset$. Since f and g are continuous, there exist $\delta_1, \delta_2 > 0$ such that $f(N^X_{\delta_1}(x)) \subseteq N^Y_\varepsilon(f(x))$ and $g(N^X_{\delta_2}(x)) \subseteq N^Y_\varepsilon(g(x))$.

Now let $\delta = \min\{\delta_1, \delta_2\}$. Then $N^X_\delta(x)$ is contained in both $N^X_{\delta_1}(x)$ and $N^X_{\delta_2}(x)$, so $f(N^X_\delta(x)) \cap g(N^X_\delta(x)) \subseteq N^Y_\varepsilon(f(x)) \cap N^Y_\varepsilon(g(x)) = \emptyset$. In particular, this means that for any $z \in N^X_\delta(x))$ we have $f(z) \neq g(z)$, so $z \in X \setminus K$. Thus, $N^X_\delta(x) \subseteq X \setminus K$, so $X \setminus K$ is open, as desired.

Problem 4: Let (X, d) be an ultrametric space. Recall that this means that (X, d) is a metric space satisfying a stronger version of the triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \text{ for all } x, y, z \in X.$$

(a) (5 pts) Fix $a \in X$ and $\varepsilon > 0$, and let

$$C_\varepsilon(a) = X \setminus N_\varepsilon(a) = \{x \in X : d(x, a) \geq \varepsilon\}.$$

Prove that $C_\varepsilon(a)$ is open in X.

(b) (4 pts) Use (a) to prove that if $|X| \geq 2$, then X is disconnected. Moreover, deduce that any subset Y of X, with $|Y| \geq 2$, is disconnected.

(c) (3 pts) Let $\mathbb{R}$ denote reals with standard metric. Now use (b) to prove that any continuous function $f : \mathbb{R} \to X$ is constant (that is, there exists $x \in X$ such that $f(t) = x$ for all $t \in \mathbb{R}$).

Solution: (a) Again we will give two solutions:

Solution 1: Let $x \in C_\varepsilon(a)$, so that $d(x, a) \geq \varepsilon$. We will show that $N_\varepsilon(x) \subseteq C_\varepsilon(a)$ (which would imply that $C_\varepsilon(a)$ is open). Indeed, take any $z \in N_\varepsilon(x)$, so $d(x, z) < \varepsilon$. By ultrametric inequality we have $\varepsilon \leq d(x, a) \leq \max\{d(x, z), d(z, a)\}$. Since $d(x, z) < \varepsilon$, this forces $d(z, a) \geq \varepsilon$, whence $z \in C_\varepsilon(a)$, as desired.

Solution 2: Here we will show that $N_\varepsilon(a) = X \setminus C_\varepsilon(a)$ is closed. Take any sequence $\{x_n\}$ in $N_\varepsilon(a)$, and suppose that $x_n \to x$ for some $x \in X$. We need to show that $x \in N_\varepsilon(a)$.

Since $x_n \to x$, we have $d(x_n, x) \to 0$, so in particular $d(x_n, x) < \varepsilon$ for some n, and by assumption $d(x_n, a) < \varepsilon$ (for all n). Therefore, by ultrametric inequality we have $d(x, a) \leq \max\{d(x_n, a), d(x_n, x)\} < \varepsilon$, so $x \in N_\varepsilon(a)$, as desired.

(b) Since $|X| \geq 2$, we can find two distinct points $x, a \in X$. Let $\varepsilon = d(x, a)$. Then $a \in N_\varepsilon(a)$ and $x \in C_\varepsilon(a)$, so $N_\varepsilon(a)$ and $C_\varepsilon(a)$ are both non-empty. Since $N_\varepsilon(a)$ is always open and $C_\varepsilon(a)$ is open by (a), we deduce that X is a disjoint union of two non-empty open sets, so X is disconnected.

If Y is any subset of X with $|Y| \geq 2$, then Y itself is an ultrametric space, so applying the result of the previous paragraph to Y, we deduce that Y is also disconnected.

(c) Since $\mathbb{R}$ is connected and f is continuous, $f(\mathbb{R})$ must also be connected. Since $f(\mathbb{R}) \subseteq X$ and X is disconnected, part (b) implies that $|f(\mathbb{R})| = 1$, which is equivalent to saying that f is constant.