Solutions to Homework #9

1. Let $a, b \in \mathbb{R}$ with $a < b$, and let $\{f_n\}$ be a sequence of differentiable functions from $[a, b]$ to $\mathbb{R}$. Suppose that both the sequences $\{f_n\}$ and $\{f_n'\}$ are uniformly bounded. Prove that the sequence $\{f_n\}$ is equicontinuous (and hence has a uniformly convergent subsequence).

Solution: We are given that there exists $M \in \mathbb{R}$ such that $|f_n'(x)| \leq M$ for all $x \in [a, b]$ and $n \in \mathbb{N}$. Hence by the Mean Value Theorem, we have $|f_n(x) - f_n(y)| \leq M|x - y|$ for all $n \in \mathbb{N}$ and $x, y \in [a, b]$. From this inequality it is clear that $\{f_n\}$ is equicontinuous (the definition of equicontinuity holds with $\delta = \frac{\varepsilon}{M}$). Since $[a, b]$ is compact and $\{f_n\}$ is also uniformly bounded, Arzela-Ascoli Theorem implies that $\{f_n\}$ has a uniformly convergent subsequence.

2. The goal of this problem is to show that the statement of the Arzela-Ascoli theorem may be false if the domain is not totally bounded.

(a) Consider functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \begin{cases} \frac{|x|}{n} & \text{if } |x| \leq n \\ 1 & \text{if } |x| > n \end{cases}$

Prove that the sequence $\{f_n\}$ is uniformly bounded and equicontinuous, but does not have a uniformly convergent subsequence. Deduce that Arzela-Ascoli Theorem does not hold for $X = \mathbb{R}$.

(b)* (bonus) Now let (X, d) be any unbounded metric space. Show that there exists a sequence of continuous functions $f_n : X \to \mathbb{R}$ which is uniformly bounded and equicontinuous, but does not have a uniformly convergent subsequence.

Solution: (a) Clearly, $|f_n(x)| \leq 1$ for all n and x, so $\{f_n\}$ is uniformly bounded. Also, a straightforward case-by-case analysis shows that $|f_n(x) - f_n(y)| \leq \frac{|x-y|}{n} \leq |x - y|$ for all n (alternatively see the proof of (b) below), so $\{f_n\}$ is equicontinuous (with $\delta = \varepsilon$). It is also clear that $f_n \to 0$ pointwise.

Suppose now that f_n has a uniformly convergent subsequence $\{f_{n_k}\}$. Then $f_{n_k} \rightrightarrows 0$, so there exists $K \in \mathbb{N}$ such that $|f_{n_k}(x)| = |f_{n_k}(x) - 0| < \frac{1}{2}$ for all $k \geq K$ and $x \in \mathbb{R}$. This clearly cannot happen since $f_{n_k}(n_k) = 1$ for all k.

(b) Fix $a \in X$. Since X is unbounded, we can find a sequence $\{x_n\}$ in X such that $d(x_n, a) \geq n$ for all n.

Define the functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \begin{cases} \frac{d(x,a)}{n} & \text{if } d(x, a) \leq n \\ 1 & \text{if } d(x, a) > n \end{cases}$

Note that the sequence $\{f_n\}$ from part (a) is a special case of this construction where $X = \mathbb{R}$ and $a = 0$.

We claim that $\{f_n\}$ has required properties. The proof is very similar to part (a); the only thing we will check explicitly is why $\{f_n\}$ is equicontinuous. We will show that $|f_n(x)-f_n(y)| \leq \frac{d(x,y)}{n} \leq d(x,y)$ for all $n \in \mathbb{N}$ and $x, y \in X$ (hence definition of equicontinuity holds with $\delta = \varepsilon$).

So take any $x, y \in X$ and $n \in \mathbb{N}$. If $f_n(x) = f_n(y)$, there is nothing to prove, so assume that $f_n(x) \neq f_n(y)$. WOLOG $f_n(x) < f_n(y)$. Since the values of f_n are bounded above by 1, we have $f_n(x) < 1$, whence (based on the formula for f_n), $f_n(x) = \frac{d(x,a)}{n}$. It is also clear that $f_n(y) \leq \frac{d(y,a)}{n}$ (this is true for any y and any n). Therefore,

$$|f_n(y) - f_n(x)| = f_n(y) - f_n(x) \leq \frac{d(y,a) - d(x,a)}{n} \leq \frac{d(x,y)}{n} \leq d(x,y),$$

where the next-to-last step holds by triangle inequality.

3. Pugh, problem 9 on p. 264.

Solution: We claim that the family $\{f_n\}$ is equicontinuous $\iff$ f is constant. The backwards direction is clear. We prove the forward direction by contrapositive. Suppose that $\{f_n\}$ is equicontinuous but f is not constant and choose $x, y \in \mathbb{R}$ with $f(y) \neq f(x)$. Let $\varepsilon = |f(y) - f(x)|$. By equicontinuity there exists $\delta > 0$ such that $|u - v| < \delta$ implies $|f_n(u) - f_n(v)| < \varepsilon$ for all $u, v \in \mathbb{R}$. Choose n such that $\frac{|y-x|}{n} < \delta$ and let $u = \frac{y}{n}$ and $v = \frac{x}{n}$. Then $|u - v| < \delta$ but $|f_n(u) - f_n(v)| = |f(nu) - f(nv)| = |f(y) - f(x)| = \varepsilon$, a contradiction.

4. Pugh, problem 14 on p. 264

Solution: We will use the following terminology: Given points $x, y \in M$ and $\delta > 0$, a δ-chain from x to y is a finite sequence $x = x_0, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) < \delta$ for all i. Thus M is chain-connected if and only if for any $x, y \in M$ and any $\delta > 0$ there exists a δ-chain from x to y in M.

"$\Rightarrow$" Suppose that M is chain-connected and $\mathcal{F}$ is an equicontinuous family of functions from M to $\mathbb{R}$. Choose $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f(x) - f(y)| < 1$ for all $f \in \mathcal{F}$.

Suppose now that $\mathcal{F}$ is bounded at some (fixed) point $p \in M$, so there exists $C \in \mathbb{R}$ such that $|f(p)| \leq C$ for all $f \in \mathcal{F}$. If $q \in M$ is such that $d(q, p) < \varepsilon$, then for any $f \in \mathcal{F}$ we have $|f(q)| = |f(q) - f(p) + f(p)| \leq |f(q) - f(p)| + |f(p)| \leq C + 1$, so $\mathcal{F}$ is bounded at q as well. Applying this argument several times, we see that $\mathcal{F}$ is bounded at any point $x \in M$ such that there exists a δ-chain from p to x. But M is chain-connected, so such a chain exists for any $x \in M$. It follows that $\mathcal{F}$ is bounded at every $x \in M$, that is, $\mathcal{F}$ is pointwise bounded on M.

"$\Rightarrow$" We argue by contrapositive. Suppose M is not chain connected, so there exists $\delta > 0$ and $a, b \in M$ such that there is no δ-chain from a to b.

Define a relation $\sim$ on M by $x \sim y \iff$ there is a δ-chain from x to y. We claim this is an equivalence relation. Indeed, $x \sim x$ since x, x is a δ-chain from x to itself. If $x \sim y$ and $x = x_0, x_1, \ldots, x_n = y$ is a δ-chain from x to y, then $x_n, x_{n-1}, \ldots, x_0$ is a δ-chain from y to x. Finally if $x \sim y$ and $y \sim z$, there exist δ-chains $x = x_0, \ldots, x_n = y$ and $y = y_0, \ldots, y_m = z$; then $x_0, \ldots, x_n = y_0, \ldots, y_m$ is a δ-chain from x to z, so $x \sim z$.

By assumption, there exist non-equivalent elements $a, b \in M$, so there is more than one equivalence class with respect to $\sim$. Let A be one of the equivalence classes and let $B = M \setminus A$, so that $M = A \sqcup B$. Note that by construction both A and B are non-empty.

If $a \in A$ and $x \in M$ are such that $d(a, x) < \delta$, then a, x is a δ-chain from a to x, so $x \sim a$ and hence $x \in A$ (since A is the equivalence class of any of its elements), so $x \notin B$. By contrapositive,

$$\text{for any } a \in A, b \in B \text{ we must have } d(a, b) \geq \delta \qquad (***)$$

Now define a sequence of functions $\{f_n : X \to \mathbb{R}\}$ by setting

$$f_n(x) = \begin{cases} n & \text{if } x \in A \\ 0 & \text{if } x \in B. \end{cases}$$

Clearly, $\{f_n\}$ is bounded at any $x \in B$ and not bounded at any $x \in A$ (recall that we made sure that A and B are non-empty). It remains to show that $\{f_n\}$ is equicontinuous. Given any $\varepsilon > 0$, let us use δ introduced above. If $d(x, y) < \delta$, then by (***) we must have $x, y \in A$ or $x, y \in B$. But each f_n is constant on both A and B, so $|f_n(x) - f_n(y)| = 0 < \delta$.

5. Let (X, d) be a metric space, and let $(B(X), d_{unif})$ be the metric space of all bounded functions $f : X \to \mathbb{R}$ with uniform metric:

$$d_{unif}(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Let $\mathcal{F} \subseteq B(X)$. Let (P) be one of the three properties: pointwise bounded, uniformly bounded and equicontinuous. Prove that if $\mathcal{F}$ has (P), then its closure $\overline{\mathcal{F}}$ also has (P). (You need to give three different proofs, one for each property).

Solution: (1) Suppose that $\mathcal{F}$ is pointwise bounded. Thus for any $x \in X$ there exists $C_x \in \mathbb{R}$ such that $|f(x)| \leq C_x$ for all $f \in \mathcal{F}$.

Now take any $h \in \overline{\mathcal{F}}$. There exists $f \in \mathcal{F}$ with $d_{unif}(h, f) < 1$, so in particular $|h(x) - f(x)| < 1$. Then $|h(x)| < |f(x)| + |h(x) - f(x)| \leq C_x + 1$. Since the obtained bound does not depend on h and since $x \in X$ was arbitrary, we deduce that $\overline{\mathcal{F}}$ is pointwise bounded on X.

(2) Preservation of uniform boundedness is very similar to (1).

(3) Suppose now that $\mathcal{F}$ is equicontinuous. Take any $\varepsilon > 0$, and let $\delta > 0$ be such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{3}$ for all $f \in \mathcal{F}$. We claim that $d(x, y) < \delta$ implies that $|h(x) - h(y)| < \varepsilon$ for all $h \in \overline{\mathcal{F}}$.

Indeed, given $h \in \overline{\mathcal{F}}$, we can find $f \in \mathcal{F}$ with $d_{unif}(h, f) < \frac{\varepsilon}{3}$. Then $|h(t) - f(t)| < \frac{\varepsilon}{3}$ for all $t \in X$, so if $d(x, y) < \delta$, we have

$$|h(x) - h(y)| \leq |h(x) - f(x)| + |f(x) - f(y)| + |f(y) - h(y)| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$