Solutions to Homework #8

1. Let X be a metric space and $\{f_n\}, f$ functions from X to $\mathbb{R}$. Suppose that $f_n \rightrightarrows f$ on X.

(i) Prove that if each f_n is bounded, then f is bounded.

(ii) Assume that f is bounded. Prove that there exists $M \in \mathbb{N}$ and $C \in \mathbb{R}$ such that $|f_n(x)| \leq C$ for all $n \geq M$ and $x \in X$. In other words, prove that the sequence $\{f_n\}$ becomes uniformly bounded after we remove the first few terms at the beginning.

Solution: (i) Since $f_n \rightrightarrows f$, there exists $M \in \mathbb{N}$ such that $|f_n(x) - f(x)| < 1$ for all $n \geq M$ and all $x \in X$. We are given that each f_n is bounded; in particular, f_M is bounded, so there exists $C \in \mathbb{R}$ such that $|f_M(x)| \leq C$ for all $x \in X$. Then $|f(x)| = |f(x) - f_M(x) + f_M(x)| \leq |f(x) - f_M(x)| + |f_M(x)| < C + 1$ for all $x \in X$, so f is bounded.

(ii) As in (i) there exists $M \in \mathbb{N}$ such that $|f_n(x) - f(x)| < 1$ for all $n \geq M$ and all $x \in X$. This time we are given that f is bounded, so there exists $D \in \mathbb{R}$ such that $|f(x)| \leq D$ for all $x \in X$. Then for all $n \geq M$ and $x \in X$ we have $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < D + 1$, so the assertion of (ii) holds with $D = C + 1$.

2. Problem 5 on p. 263 in Pugh (see Exercise 3.36 for the definition of jump and removable discontinuities). A clarification on the statement: in each part of the problem you are given some property (P) of functions; the question is the following: if each f_n has property (P), is it always true that the limiting function f also has (P). In part (e) countable should mean 'infinite countable'.

Solution: (a) true by Theorem 14.1 from class.

(b) True. We first prove a lemma:

Lemma: *If $f_n \rightrightarrows f$ on some metric space X and $x \in X$ is such that f_n is continuous at x for infinitely many n, then f is continuous at x.*

Proof. By assumption we can find an infinite sequence $n_1 < n_2 < \dots$ such that each f_{n_k} is continuous at x. But then $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$, and since $f_n \rightrightarrows f$, it is clear that $f_{n_k} \rightrightarrows f$ as well. Since each f_{n_k} is continuous at x, Theorem 1 from 4.1 in Pugh implies that f is continuous at x. □

We can now prove that (b) is true by contrapositive. Suppose that f has at least 11 points of discontinuity, call them $x_1, \dots, x_{11}$. For each $1 \leq i \leq 11$

let $S_i = \{n \in \mathbb{N} : f_n \text{ is continuous at } x_i\}$. By (the contrapositive of) the Lemma, each set S_i is finite, and hence $\bigcup_{i=1}^{11} S_i$ is finite. Thus, there exists $n \in \mathbb{N}$ which does not lie in S_i for any i, but then f_n must be discontinuous at each of the 11 points $x_1, \ldots, x_{11}$, a contradiction.

(c) False. Let $[a, b] = [0, 10]$, let $g : [0, 10] \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases}$$

Define $f_n(x) = \frac{g(x)}{n}$ for each n. Then $f_n \rightrightarrows 0$ since $|f_n(x)| \le \frac{1}{n}$ and $\frac{1}{n}$ does not involve x and tends to 0. Clearly each f_n has 11 discontinuities, namely $0, 1, \ldots, 10$, but $f = 0$ has no discontinuities.

(d) False. Let $[a, b] = [0, 1]$, let $g : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x = \frac{1}{k} \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Now define

$$f_n(x) = \begin{cases} f(x) & \text{if } x \ge \frac{1}{n} \\ 0 & \text{if } x < \frac{1}{n} \end{cases}$$

Then $f_n \rightrightarrows f$ since $f_n = f$ on the interval $[\frac{1}{n}, 1]$ and $|f_n - f| \le \frac{1}{n}$ on $[0, \frac{1}{n})$, each f_n has exactly n discontinuities (at the points $\frac{1}{k}$ with $k \le n$), but f has infinitely many discontinuities.

(e) False. Let $[a, b] = [0, 1]$, let $g : [0, 1] \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{k} \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Define $f_n(x) = \frac{g(x)}{n}$ for each n. Then $f_n \rightrightarrows 0$ as in (c), each f_n has infinitely many discontinuities, all of them of jump type, but f has no discontinuities.

(f) False. Let $[a, b] = [-1, 1]$, and define $f_n : [-1, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 + \frac{1}{n} & \text{if } x \ge 0 \text{ and } x \in \mathbb{Q} \\ 1 & \text{if } x \ge 0 \text{ and } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x < 0 \text{ and } x \in \mathbb{Q} \\ 0 & \text{if } x < 0 \text{ and } x \notin \mathbb{Q} \end{cases}$$

It is easy to show that f_n has an oscillating discontinuity at every x and in particular has no jump discontinuities. On the other hand, $f_n \rightrightarrows f$ where

$$f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0, \end{cases}$$

and clearly f has a jump discontinuity at 0.

(g) True. To prove this it suffices to show the following: if $f_n \rightrightarrows f$ and for each $c \in (a,b]$ the left-hand limit $\lim_{x\to c^-} f_n(x)$ exists, then $\lim_{x\to c^-} f(x)$ also exists, and similarly for the right-hand limit at each $c \in [a,b)$.

Let us fix $c \in (a,b]$. We first prove that the sequence of left-hand limits $\{L_n = \lim_{x\to c^-} f_n(x)\}_{n=1}^{\infty}$ is Cauchy. Take any $\varepsilon > 0$. Since $f_n \rightrightarrows f$ and hence $\{f_n\}$ is uniformly Cauchy, there exists $M \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \frac{\varepsilon}{3}$ for all $n, m \geq M$ and all $x \in [a,b]$.

Now take any $n, m \geq M$. Since $\lim_{x\to c^-} f_n(x) = L_n$, there exists $\delta_n > 0$ such that $|f_n(x) - L_n| < \frac{\varepsilon}{3}$ for all $x \in (c - \delta_n, c)$. Similarly, there exists $\delta_m > 0$ such that $|f_m(x) - L_m| < \frac{\varepsilon}{3}$ for all $x \in (c - \delta_m, c)$. Let $\delta = \min\{\delta_n, \delta_m\}$ and choose any $x \in (c - \delta, c)$. Then

$$|L_n - L_m| \leq |L_n - f_n(x)| + |f_n(x) - f_m(x)| + |f_m(x) - L_m| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Thus, the sequence $\{L_n\}$ is indeed Cauchy and hence converges to some $L \in \mathbb{R}$. We now prove that $\lim_{x\to c^-} f(x) = L$. Again take any $\varepsilon > 0$. Since $f_n \rightrightarrows f$, there exists $N_1 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geq N_1$ and $x \in [a,b]$. Since $L_n \to L$, there exists $N_2 \in \mathbb{N}$ such that $|L_n - L| < \frac{\varepsilon}{3}$ for all $n \geq N_2$. Finally, choose any $n \geq \max\{N_1, N_2\}$. Since $\lim_{x\to c^-} f_n(x) = L_n$, there exists $\delta > 0$ such that $|f_n(x) - L_n| < \frac{\varepsilon}{3}$ for all $x \in (c - \delta, c)$. Then for all $x \in (c - \delta, c)$ we have

$$|f(x) - L| \leq |f(x) - f_n(x)| + |f_n(x) - L_n| + |L_n - L| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon,$$

so $\lim_{x\to c^-} f(x) = L$, as desired.

3. For each $\alpha \in \mathbb{R}$ define the function $I_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$I_\alpha(x) = \begin{cases} 0 & \text{if } x < \alpha \\ 1 & \text{if } x \geq \alpha \end{cases}$$

Now let $S = \{s_1, s_2, \ldots\}$ be a countable infinite subset of $\mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} \frac{I_{s_n}(x)}{2^n}$. Prove that

(a) the series always converges (so that f is indeed defined on $\mathbb{R}$),

(b) f is increasing (that is, $x < y$ implies $f(x) \leq f(y)$), and

(c)* f is continuous at $x \iff x \notin S$.

Solution: (a) Let $f_n(x) = \frac{I_{s_n}(x)}{2^n}$, so that $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Clearly, $|f_n(x)| \leq \frac{1}{2^n}$. Since the series $\sum \frac{1}{2^n}$ converges, by the Weierstrass M-test, $\sum_{n=1}^{\infty} f_n$ converges uniformly; in particular, $f(x)$ is always defined.

(b) Take any real numbers $x < y$. Then $f(y) - f(x) = \sum_{n=1}^{\infty}(f_n(y) - f_n(x))$ (note that the series on the right is convergent as the difference of two convergent series). It is clear from the definition that each f_n is increasing, so $f_n(y) - f_n(x) \geq 0$. Thus, $f(y) - f(x)$ is the sum of a (convergent) series with non-negative terms, so $f(y) - f(x) \geq 0$, as desired.

(c) Suppose first that $x \notin S$. Then each f_n is continuous at x, whence all the partial sums $s_n = \sum_{i=1}^{n} f_i$ of the functional series $\sum f_i$ are continuous at x. Since $s_n \rightrightarrows f$ as we showed above, Theorem 1 from 4.1 in Pugh implies that f is continuous at x.

Suppose now that $x \in S$, so $x = s_n$ for some n. We can write $f = g + h$ where $g = f_n$ and $h = \sum_{m \neq n} f_m$. Then we can apply the above argument to h (and the set $S \setminus \{s_n\}$) to conclude that h is continuous at $x = s_n$. On the other hand, $g = f_n$ is clearly discontinuous at s_n. Hence f must also be discontinuous at s_n (if f were continuous at s_n, then $g = f - h$ would also be continuous at s_n as the difference of two functions continuous at s_n).

4. Problem 7.3:15 from Bergman's supplement to Rudin (page 79), see

`http://math.berkeley.edu/~gbergman/ug.hndts/m104_Rudin_exs.pdf`

You can assume that the functions are real-valued (not complex-valued); also J denotes the natural numbers.

Solution: We will use notations from Bergman's notes except that we will denote the set of natural numbers by $\mathbb{N}$ (as usual) instead of J.

(a) First note that $d((p,s),(p',s')) \leq 1$ for all $(p,s),(p',s') \in X$. Consider any three points $(p_1,s_1),(p_2,s_2),(p_3,s_3) \in X$. If $p_1 \neq p_2$ or $p_2 \neq p_3$, then $d((p_1,s_1),(p_2,s_2)) + d((p_2,s_2),(p_3,s_3)) \geq 1 \geq d((p_1,s_1),(p_3,s_3))$.

And if $p_1 = p_2 = p_3$, then $d((p_1,s_1),(p_3,s_3)) = |s_1 - s_3| \leq |s_1 - s_2| + |s_2 - s_3| = d((p_1,s_1),(p_2,s_2)) + d((p_2,s_2),(p_3,s_3))$.

(b) We start with a lemma:

Lemma: *Let $x = (p, \frac{1}{n}) \in X$ (where $n \in \mathbb{N}$), and let $\delta < \frac{1}{n(n+1)}$. Then $N_\delta^X(x) = x$ (so in particular, x is an isolated point of X).*

Proof: Take any $y \in X$ with $d(x,y) < \delta$. Since $\delta < \frac{1}{n} \leq 1$, y must have the form $y = (p, \frac{1}{m})$ (that is, the first coordinates of x and y coincide and the second coordinate of y is nonzero). We need to show that $m = n$.

If $m \neq n$, then $m \leq n-1$ or $m \geq n+1$. If $m \leq n-1$, then $d(x,y) = \frac{1}{m} - \frac{1}{n} \leq \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} > \delta$, a contradiction. And if $m \geq n+1$, then $d(x,y) = \frac{1}{n} - \frac{1}{m} \geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > \delta$, a contradiction. Thus, $m = n$.

□

We proceed with the proof of (b). We first show that $f_n \to f$ pointwise $\iff$ F is continuous.

"$\Rightarrow$" Clearly, any function will be continuous at an isolated point of a metric space. Thus, by Lemma we only need to show that F is continuous at points of the form $(p, 0)$. Fix $p \in E$ and $\varepsilon > 0$. Since $f_n \to f$ pointwise, there exists $N \in \mathbb{N}$ such that $|f_n(p) - f(p)| < \varepsilon$ for all $n \geq N$. Now pick any $0 < \delta < \frac{1}{N}$. It is clear that $N_\delta^X((p, 0))$ is contained in the set $\{(p, \frac{1}{n}) : n \geq N\}$. But if $n \geq N$, then by definition of F we have

$$|F((p, \tfrac{1}{n})) - F((p, 0))| = |f_n(p) - f(p)| < \varepsilon, \text{ so } F((p, \tfrac{1}{n})) \in N_\varepsilon^{\mathbb{R}}(F((p, 0))).$$

Thus, we showed that $F(N_\delta^X((p, 0))) \subseteq N_\varepsilon^{\mathbb{R}}(F((p, 0)))$, so F is continuous at $(p, 0)$.

The proof of the reverse implication '$\Leftarrow$"is almost the same. Assume that F is continuous. Hence, for any $p \in E$ and $\varepsilon > 0$, we can find $\delta > 0$ such that $F(N_\delta^X((p, 0))) \subseteq N_\varepsilon^{\mathbb{R}}(F((p, 0)))$. Now pick $N \in \mathbb{N}$ with $N > \frac{1}{\delta}$. The same computation as above then shows that $|f_n(p) - f(p)| < \varepsilon$ for all $n \geq N$, so $f_n \to f$ pointwise.

Now we prove that $f_n \rightrightarrows f \iff F$ is uniformly continuous.

'$\Rightarrow$" Fix $\varepsilon > 0$. By the definition of uniform convergence and the Cauchy criterion for uniform convergence, there exists $N \in \mathbb{N}$ such that $|f_n(p) - f(p)| < \varepsilon$ for all $n \geq N$ and all $p \in E$ and $|f_n(p) - f_m(p)| < \varepsilon$ for all $n, m \geq N$ and all $p \in E$.

Pick $0 < \delta < \frac{1}{N(N+1)}$, and take any two distinct points $x, y \in X$ with $d(x, y) < \delta$. Since $\delta < 1$, x and y must have the same first coordinate, so $x = (p, s)$ and $y = (p, s')$ for some $s, s' \in S$. If s and s' are both nonzero, then Lemma implies that $s = \frac{1}{n}$ and $s' = \frac{1}{m}$ with $n, m > N$ (if for instance, $n < N$, then $\delta < \frac{1}{N(N+1)} < \frac{1}{n(n+1)}$, so by Lemma $N_\delta^X(x) = x$, a contradiction since $y \in N_\delta^X(x)$). Therefore, we have

$$|F(x) - F(y)| = |f((p, \tfrac{1}{n})) - f((p, \tfrac{1}{m}))| = |f_n(p) - f_m(p)| < \varepsilon.$$

Suppose now that $s = 0$ or $s' = 0$. Since $x \neq y$, we cannot have $s = s' = 0$, and WOLOG assume that $s' = 0$, so that $s = \frac{1}{n}$ for some $n \in \mathbb{N}$. Then $\frac{1}{n} = d(x, y) < \delta < \frac{1}{N}$, so $n > N$, and therefore,

$$|F(x) - F(y)| = |f((p, \tfrac{1}{n})) - f((p, 0)| = |f_n(p) - f(p)| < \varepsilon.$$

Thus, inequality $d(x, y) < \delta$ always implies $|F(x) - F(y)| < \varepsilon$, whence F is uniformly continuous.

"$\Leftarrow$" Now suppose that F is uniformly continous. Thus, given $\varepsilon > 0$, there is $\delta > 0$ such that $d(x, y) < \delta$ always implies $|F(x) - F(y)| < \varepsilon$. Pick any $N \in \mathbb{N}$ with $\frac{1}{N} < \delta$. Then for any $p \in E$ and any $n \geq N$ we have $d((p, \frac{1}{n}), (p, 0)) < \delta$, so $|F((p, \frac{1}{n})) - F((p, 0))| < \varepsilon$. Since $F((p, \frac{1}{n})) - F((p, 0)) = f_n(p) - f(p)$, by definition $f_n \rightrightarrows f$.