Solutions to Homework #7

1. Let X be a metric space and Y a subset of X.

(a) Prove that if X is complete and Y is closed in X, then Y is complete.
(b) Prove that if Y is complete, then Y is closed in X.

Recall that we proved the analogous statements with 'complete' replaced by 'sequentially compact' (Theorem 9.2 and Theorem 8.1, respectively).

Solution: (a) Take any Cauchy sequence $\{y_n\}$ in Y. Then $\{y_n\}$ is also a sequence in X, so (since X is complete) $\{y_n\}$ converges to some $x \in X$. Then $x \in \overline{Y}$ by Lemma 6.2 (sequential characterization of closures), and since Y is closed, we have $x \in Y$. Thus, every Cauchy sequence in Y converges in Y, so Y is complete.

(b) We will prove that Y is closed by checking the inclusion $\overline{Y} \subseteq Y$.

So take any $x \in \overline{Y}$. By Lemma 6.2, there is a sequence $\{y_n\}$ in Y which converges to x. Since a convergent sequence in any metric space is Cauchy, $\{y_n\}$ is a Cauchy sequence (as a sequence in X, hence also as a sequence in Y). Since Y is complete, $\{y_n\}$ must converge in Y. Finally, since a sequence has at most one limit, we conclude that $x \in Y$. Thus, $\overline{Y} \subseteq Y$ as desired.

2. This problem describes a fancy way to show that closed bounded intervals in $\mathbb{R}$ are connected. A metric space (X, d) is called *chain-connected* if for any $x, y \in X$ and $\delta > 0$ there exists a finite sequence $x_0, x_1, \ldots, x_n$ of points of X such that $x_0 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \delta$ for all i.

(a) Let X be metric space which is compact and chain-connected. Prove that X is connected.
(b) Prove that a closed bounded interval $[a, b] \subseteq \mathbb{R}$ is chain-connected and deduce from (a) that $[a, b]$ is connected.

Solution: (a) Suppose that X is disconnected. Then by Problem 2 in HW#6, there exists a continuous function $f : X \to \mathbb{R}$ with $f(X) = \{-1, 1\}$. Since X is compact, by Theorem 11.4 from class, f must be uniformly continuous. We will now show that uniform continuity of such f contradicts the chain property.

Indeed, since f is uniformly continuous, there must exist $\delta > 0$ such that $|f(x) - f(y)| < 2$ for any $x, y \in X$ with $d(x, y) < \delta$. Since $f(X) = \{-1, 1\}$, the inequality $|f(x) - f(y)| < 2$ is only possible if $f(x) = f(y)$. Thus, $d(x, y) < \delta$ forces $f(x) = f(y)$.

Now choose any $x, y \in X$ with $f(x) = -1$ and $f(y) = 1$. By the chain property there exists a finite sequence $x_0, x_1, \ldots, x_n$ of points of X such that $x_0 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \delta$ for all i. Hence we must have $f(x_i) = f(x_{i+1})$ for all i, whence $f(x) = f(x_0) = f(x_1) = \ldots = f(x_n) = f(y)$, which is a contradiction.

(b) Let $x, y \in [a, b]$; WOLOG $x \leq y$. Given $\delta > 0$, choose $n \in \mathbb{N}$ such that $n > \frac{y-x}{\delta}$, and define $x_i = x + i\frac{y-x}{n}$. Then it is clear that the sequence $x_0 = x, x_1, \ldots, x_n = y$ satisfies the definition of the chain property.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

(a) Assume that f' is bounded, that is, there exists $M \in \mathbb{R}$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.
(b) Now assume that $f'(x) \to \infty$ as $x \to \infty$. Prove that f is not uniformly continuous.

Solution: (a) By the mean value theorem, for any $x, y \in \mathbb{R}$, with $x < y$, there exists $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. By assumption, $|f'(c)| \leq M$, so $|f(y) - f(x)| \leq M|y - x|$.

Now given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$. Then $|y - x| < \delta$ implies that $|f(y) - f(x)| < M\delta = \varepsilon$, so f is uniformly continuous.

(b) To prove that f is not uniformly continuous, we need to show that there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exist $x, y \in \mathbb{R}$ with $|y - x| < \delta$ and $|f(y) - f(x)| \geq \varepsilon$. We will show that this is true for $\varepsilon = 1$, but in fact, we could use any ε.

Fix $\delta > 0$. Since $f'(t) \to \infty$ as $t \to \infty$, there exists $N = N(\delta)$ such that $f'(t) \geq \frac{2}{\delta}$ for all $t \geq N$. Now take $x = N$ and $y = x + \frac{\delta}{2}$; then $|y - x| = \frac{\delta}{2} < \delta$. On the other hand, by the mean value theorem, $|f(y) - f(x)| = |y - x||f'(c)|$ for some $c \in (x, y)$. Since $c > x = N$, we have $f'(c) \geq \frac{2}{\delta}$, so $|f(y) - f(x)| \geq 1$, as desired.

4. The goal of this problem is to fill in the details of the construction of the completion of a metric space discussed in Lecture 13. Part (a) below is Claim 1 from class; (b) and (c) form Claim 2 from class, and (d) is Claim 3 from class.

We start by recalling the notations introduced in class. Let (X, d) be a metric space. Let $\Omega = \Omega(X)$ be the set of all Cauchy sequences $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in X$ for each n. Define the relation $\sim$ on Ω by setting

$$\{x_n\} \sim \{y_n\} \iff \lim_{n \to \infty} d(x_n, y_n) = 0.$$

(a) Prove that $\sim$ is an equivalence relation.

Now let $\widehat{X} = \Omega/\sim$, the set of equivalence classes with respect to $\sim$. The equivalence class of a sequence $\{x_n\}$ will be denoted by $[x_n]$. For instance, $[\frac{1}{n}] = [\frac{1}{n^2}]$ since the sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$ are equivalent. Given an element $x \in X$, we will denote by $[x] \in \widehat{X}$ the equivalence class of the constant sequence all of whose elements are equal to x.

Now define the function $D : \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0}$ by setting

$$D([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n) \qquad (***)$$

(b) Prove that the limit on the right-hand side of (***) always exists and that the function D is well-defined (that is, if $[x_n] = [x'_n]$ and $[y_n] = [y'_n]$, then $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$).

(c) Prove that $(\widehat{X}, D)$ is a metric space

(d) Consider the map $\iota : X \to \widehat{X}$ given by $\iota(x) = [x]$ (that is, ι sends each x to the equivalence class of the corresponding constant sequence). Prove that ι is injective and $D(\iota(x), \iota(y)) = d(x, y)$ for all $x, y \in X$. This implies that (X, d) is isometric to the metric space $(\iota(X), D)$ (so identifying X with $\iota(X)$, we can think of X as a subset of $(\widehat{X}, D)$).

Solution: (a) Reflexivity: $\{x_n\} \sim \{x_n\}$ since $\lim_{n\to\infty} d(x_n, x_n) = 0$. Symmetry ($\{x_n\} \sim \{y_n\} \Rightarrow \{y_n\} \sim \{x_n\}$) follows from the fact that d is symmetric ($d(x, y) = d(y, x)$).

Finally, transitivity follows from the triangle inequality: if $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$. Since $0 \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$, by the squeeze theorem we have $\lim_{n\to\infty} d(x_n, y_n) = 0$, so $\{x_n\} \sim \{z_n\}$.

(b) First we show that for any Cauchy sequences $\{x_n\}$ and $\{y_n\}$, the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists. Since $\mathbb{R}$ is complete, it suffices to show that the sequence $\{d(x_n, y_n)\}$ is Cauchy.

Fix $\varepsilon > 0$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy, there exists $M_1, M_2 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for all $n, m \geq M_1$ and $d(y_n, y_m) < \frac{\varepsilon}{2}$ for all $n, m \geq M_2$.

Let $M = \max\{M_1, M_2\}$. Then for all $n, m \geq M$ we have

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < d(x_m, y_m) + \varepsilon,$$

and similarly, $d(x_m, y_m) < d(x_n, y_n) + \varepsilon$. Hence $|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon$ for all $n, m \geq M$, and therefore $\{d(x_n, y_n)\}$ is Cauchy.

Now we prove independence on the equivalence class. Suppose that $\{x_n\} \sim \{x'_n\}$ and $\{y_n\} \sim \{y'_n\}$. Then

$$d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n). \qquad (***)$$

Since $\lim_{n\to\infty} d(x'_n, x_n) = \lim_{n\to\infty} d(y_n, y'_n) = 0$, taking limits on both sides of (***), we conclude that $\lim_{n\to\infty} d(x'_n, y'_n) \le \lim_{n\to\infty} d(x_n, y_n)$. By the same argument, the reverse inequality also holds: $\lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} d(x'_n, y'_n)$, so $\lim_{n\to\infty} d(x'_n, y'_n) = \lim_{n\to\infty} d(x_n, y_n)$.

(c) The first two conditions in the definition of a metric space are clear, so we only need to check triangle inequality. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be any Cauchy sequences. We know that $d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$ for each n, and each of the sequences $\{d(x_n, z_n)\}, \{d(x_n, y_n)\}$ and $\{d(y_n, z_n)\}$ converges, so passing to the limit in the above inequality, we get

$$D([x_n], [z_n]) = \lim_{n\to\infty} d(x_n, z_n) \le \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n)$$
$$= D([x_n], [y_n]) + D([y_n], [z_n]).$$

(d) By definition $D(\iota(x), \iota(y))$ is the limit of the constant sequence $d(x, y)$, so $D(\iota(x), \iota(y)) = d(x, y)$. Since $d(x, y) \ne 0$ for $x \ne y$, we also conclude that $\iota(x) \ne \iota(y)$ for $x \ne y$, so ι is injective.

5. Consider functions $f_n : \mathbb{R}_{\ge 0} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{nx+1}$. Let $0 \le a \le b$ be real numbers. Prove that $\{f_n\}$ converges uniformly on $[a, b] \iff a > 0$ or $a = b = 0$.

Solution: Define $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ by $f(0) = 1$ and $f(x) = 0$ for $x > 0$. It is clear that $f_n \to f$ pointwise.

Case 1: $a = 0$ and $b > 0$. In this case f is discontinuous on $[a, b] = [0, b]$. Since each f_n is continuous on $[a, b]$, $\{f_n\}$ cannot converge uniformly to f by Theorem 14.1 from class.

Here is a proof directly from definition. Suppose, by way of contradiction, that $\{f_n\}$ converges to f uniformly. Then there exists $M \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{1}{2}$ for all $n \in M$ and all $x \in [0, b]$. Choose $n > M$ such that $\frac{1}{n} < b$, and let $x = \frac{1}{n}$. Then $x \in [0, b]$ and $f_n(x) - f(x) = \frac{1}{n \cdot \frac{1}{n} + 1} - 0 = \frac{1}{2}$, which is a contradiction.

Case 2: $a = b = 0$. Then $[a, b]$ is just one point, so pointwise convergence on $[a, b]$ is the same as uniform convergence.

Case 3: $a > 0$. Then for all $x \in [a, b]$ we have

$$|f_n(x) - f(x)| = \frac{1}{nx + 1} \le \frac{1}{na + 1}.$$

Since $\frac{1}{na+1}$ does not depend on x and converges to 0 as $n \to \infty$ (since $a > 0$), it follows that $f_n \rightrightarrows f$ on $[a, b]$.

6. Let X be a set, (Y, d) a metric space. Let $\{f_n : X \to Y\}$ be a sequence of functions, and let $f : X \to Y$ be a function.

(i) Define what it should mean for $\{f_n\}$ to converge to f uniformly and what it should mean for $\{f_n\}$ to be uniformly Cauchy (in class we gave both definitions in the case $Y = \mathbb{R}$, but there is a natural way to extend them to arbitrary Y).

(ii) Theorem 14.2 from class asserts that in the case $Y = \mathbb{R}$, a sequence $\{f_n\}$ is uniformly convergent if and only if it is uniformly Cauchy. Find a natural necessary and sufficient condition on Y under which this equivalence remain true (the answer will not depend on X as long as $X \neq \emptyset$). You do not need to write down the full proof – just state the condition and where it arises in the proof.

Solution: (i) $\{f_n\}$ converges to f uniformly if for every $\varepsilon > 0$ there exists $M = M(\varepsilon) \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq M$ and all $x \in X$.

$\{f_n\}$ is uniformly Cauchy if for every $\varepsilon > 0$ there exists $M = M(\varepsilon) \in \mathbb{N}$ such that $d(f_n(x), f_m(x)) < \varepsilon$ for all $n, m \geq M$ and all $x \in X$.

(ii) We claim that being uniformly convergent and being uniformly Cauchy are equivalent conditions $\iff$ Y is complete. First suppose that Y is not complete. Then Y has at least one sequence $\{y_n\}$ which is Cauchy but not convergent. Then the sequence of constant functions $\{f_n = y_n\}$ is uniformly Cauchy but not uniformly convergent.

If Y is complete, to prove the equivalence one essentially needs to repeat the proof of Theorem 14.2 from class. The completeness of Y is needed for the direction "$\{f_n\}$ is uniformly Cauchy $\Rightarrow$ $\{f_n\}$ is uniformly convergent". More specifically, if $\{f_n\}$ is uniformly Cauchy, then for every $x \in X$ the sequence $\{f_n(x)\}$ is a Cauchy sequence in Y, and completeness of Y enables us to define the limit function f by $f(x) = \lim_{n\to\infty} f_n(x)$.

The only other (non-obvious) modification in the proof is that instead of using the continuity of the function $x \mapsto |x|$ we need to use the fact that for any fixed $a \in Y$ the function $F : Y \to \mathbb{R}$ given by $F(y) = d(a, y)$ is continuous (this was proved in HW#3).