Solutions to Homework #6

1. Complete the proof of the backwards direction of Theorem 12.2 from class (which asserts the any interval in $\mathbb{R}$ is connected).

Solution: Let $X \subseteq \mathbb{R}$ be a closed interval.

Case 1: $X = [a, b]$. This case was done in class.

Case 2: $X = [a, b)$. This case was also done in class, but we recall the argument. Let $I = (a, b)$, and for each $\alpha \in I$ let $X_\alpha = [a, \alpha]$. Then each X_α is connected by Case 1, $X = \bigcup\limits_{\alpha \in I} X_\alpha$ and $a \in X_\alpha$ for all α, so $\cap X_\alpha \neq \emptyset$. Thus, X is connected by Problem 4.

Case 3: $X = (a, b]$. This case is analogous to Case 2.

Case 4: $X = (a, b)$. Choose $\varepsilon < \frac{b-a}{2}$, and let $Y = (a, b - \varepsilon]$ and $Z = [a + \varepsilon, b)$. Then $X = Y \cup Z$, $Y \cap Z \neq \emptyset$ and Y and Z are connected by Cases 2 and 3. Hence X is connected by Problem 4.

Case 5: $X = [a, \infty)$. In this case we write $X = \cup_{b>a}[a, b]$ and argue as in case 2.

The remaining 4 cases are all very similar to one of Cases 2-5.

2. Let X be a metric space. Prove that X is disconnected if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $f(X) = \{1, -1\}$.

Solution: "$\Rightarrow$" Suppose that $f(X) = \{1, -1\}$. Since the set $\{1, -1\}$ is clearly disconnected, by Theorem 13.1 from class, it follows that X has to be disconnected as well.

"$\Leftarrow$" Following the hint, assume that $X = A \sqcup B$ with A, B open and non-empty, and define the function $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \in B \end{cases}$$

Clearly, $f(X) = \{1, -1\}$ (note that $f(X)$ is precisely $\{1, -1\}$ and not smaller since A and B are non-empty). To prove that f is continuous, it suffices to show that the preimage of any open subset is open. We will show that in this case the preimage of ANY subset is open. This is feasible since there are very few possible preimages.

Indeed, it clear that for a subset C of $\mathbb{R}$ we have $f^{-1}(C) = X$ if C contains 1 and -1; $f^{-1}(C) = A$ if C contains 1 but not -1; $f^{-1}(C) = B$ if C contains -1 but not 1, and finally $f^{-1}(\emptyset) = \emptyset$ if C does not contain 1 or -1.

The empty set and X are always open in X, and A and B are open by assumption. Thus, $f^{-1}(C)$ is always open, and we are done.

3.

(a) Let X be a disconnected metric space, so that $X = A \sqcup B$ for some non-empty closed subsets A and B. Prove that if C is any connected subset of X, then $C \subseteq A$ or $C \subseteq B$.

(b) A metric space X is called *path-connected* if for any $x, y \in X$ there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$ (informally, this means that any two points in X can be joined by a path in X). Prove that any path-connected metric space is connected.

Solution: (a) We can always write $C = (C \cap A) \sqcup (C \cap B)$. Since A and B are closed in X, the sets $C \cap A$ and $C \cap B$ are closed in C. If C is connected, then one of those sets has to be empty. But if $C \cap A = \emptyset$, then $C \cap B = C$, that is, $C \subseteq B$. Similarly, if $C \cap B = \emptyset$, then $C \subseteq A$.

(b) We prove this by contradiction. Suppose that X is disconnected, so $X = A \sqcup B$ where A and B are closed and non-empty. Choose any $x \in A$ and $y \in B$. If X is path-connected, then there exists a continuous function $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. By Theorem 13.1, the set $C = f([0, 1])$ must be connected, so by (a), $C \subseteq A$ or $C \subseteq B$. This is impossible since C contains both x (which is not in B) and y (which is not in A).

4. Let X be a metric space, $\{X_\alpha\}_{\alpha \in I}$ a collection (not necessarily finite) of subsets of X such that $\cap_{\alpha \in I} X_\alpha$ is non-empty and $\cup_{\alpha \in I} X_\alpha = X$. Prove that if each X_α is connected, then X is connected.

Solution: We argue by contradiction. Suppose that X is disconnected, so $X = A \sqcup B$ where A and B are both closed and non-empty.

By assumption, for each $\alpha \in I$ the subset X_α is connected, so by Problem 3(a) either $X_\alpha \subseteq A$ or $X_\alpha \subseteq B$. Hence one of the following three cases must hold.

Case 1: $X_\alpha \subseteq A$ for all α. Then $X = \cup_{\alpha \in I} X_\alpha \subseteq A$, so $X = A$ and $B = \emptyset$, a contradiction.

Case 2: $X_\alpha \subseteq B$ for all α. This case is analogous to Case 1.

Case 3: there exist $\gamma, \beta \in I$ such that $X_\gamma \subseteq A$ and $X_\beta \subseteq B$. Then $X_\gamma \cap X_\beta \subseteq A \cap B = \emptyset$, which contradicts the assumption that the intersection of all X_α, $\alpha \in I$, is non-empty.

6. Let (X, d_X) and (Y, d_Y) be metric spaces, and consider the product space $X \times Y$ with metric d given by $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

(a) Prove that $(X \times Y, d)$ is indeed a metric space

(b) Prove that for every $x \in X$, the subset $\{x\} \times Y = \{(x, y) : y \in Y\}$ of $X \times Y$ is isometric to Y. Likewise for every $y \in Y$, the subset $X \times \{y\} = \{(x, y) : x \in X\}$ is isometric to X.

(c*) Prove that if X and Y are both connected, then $X \times Y$ connected.

Solution: (b) For every $x \in X$ the map $f_x : Y \to \{x\} \times Y$ given by $f_x(y) = (x, y)$ is clearly bijective and preserves distances, so $\{x\} \times Y$ is isometric to Y (and similarly $X \times \{y\}$ is isometric to X for every $y \in Y$).

(c) For every $x \in X$ and $y \in Y$ define $T_{x,y} = (\{x\} \times Y) \cup (X \times \{y\})$. By Problem 5 and part (a), $\{x\} \times Y$ and $X \times \{y\}$ are connected. Since $(\{x\} \times Y) \cap (X \times \{y\}) = \{x\} \times \{y\} \neq \emptyset$, by Problem 4 the set $T_{x,y}$ is connected.

Now fix any $y \in Y$. Note that $\cup_{x \in X} T_{x,y} = X \times Y$ (since already $\cup_{x \in X} X \times \{y\} = X \times Y$) and $\cap_{x \in X} T_{x,y} \supseteq X \times \{y\}$ is non-empty. Hence, applying Problem 4 again we conclude that $X \times Y$ is connected.

Note: A common mistake in this problem was to represent $X \times Y$ as the union of ALL sets $T_{x,y}$ (where both x and y can vary). It is true that $X \times Y$ is the union of all these sets; it is also true that the intersection of any two of these sets is non-empty. However, the intersection of ALL these sets will be empty whenever $|X| > 1$ and $|Y| > 1$, so Problem 4 cannot be applied to this collection.

7. The goal of this problem is to prove that any open subset of $\mathbb{R}$ (with standard metric) is a **disjoint** union of at most countably many open intervals.

So, let U be any open subset of $\mathbb{R}$.

(a) Define the relation $\sim$ on U by setting $x \sim y \iff x = y$ or ($x < y$ and $[x, y] \subseteq U$) or ($y < x$ and $[y, x] \subseteq U$). Prove that $\sim$ is an equivalence relation.

(b) Let A be an equivalence class with respect to $\sim$. Show that A is an open interval.

(c) Deduce from (b) that U is a disjoint union of open intervals. Then prove that the number of those intervals is at most countable.

Solution: (a) Reflexivity ($x \sim x$) and symmetry are obvious from the definition, so we only check transitivity. Suppose that $x \sim y$ and $y \sim z$ for some $x, y, z \in U$. We want to show that $x \sim z$. This is obvious if $x = y$ or $y = z$ or $x = z$, so we can assume that x, y, z are distinct.

Case 1: $x < y < z$. In this case we are given that $[x, y] \subseteq U$ and $[y, z] \subseteq U$. Since $x < y < z$, we have $[x, z] = [x, y] \cup [y, z] \subseteq U$, so $x \sim z$.

Case 2: $x < z < y$. In this case $[x, z] \subseteq [x, y]$. Since $x \sim y$, we have $[x, y] \subseteq U$ and hence $[x, z] \subseteq U$ as well.

There are 4 more cases, but each of them is analogous to Case 1 or Case 2.

(b) Let A be the equivalence class of some $x \in U$. For each $\alpha \in A$ define A_α as follows: $X_\alpha = [x, \alpha]$ if $\alpha > x$; $X_\alpha = [\alpha, x]$ if $\alpha < x$ and $X_\alpha = \{\alpha\}$ if $\alpha = x$. Note that each $X_\alpha \subseteq U$ for each α by definition of $\sim$.

We claim that

$$A = \bigcup_{\alpha \in A} X_\alpha. \qquad (***)$$

Indeed, for each $\alpha \in A$ we have $\alpha \in X_\alpha$, so $A \subseteq \bigcup_{\alpha \in A} X_\alpha$.

On the other hand, if $y \in X_\alpha$ for some $\alpha \in A$, then $y \in A$. Indeed, if $y > x$, then $y \in X_\alpha$ means that $[x, y] \subseteq X_\alpha$; hence $[x, y] \subseteq U$ by the first paragraph of the proof and so $y \in A$ (again by definition of $\sim$). The case $y < x$ is analogous. Thus, we showed each X_α is contained in A, which proves the opposite inclusion.

Each X_α is an interval, hence connected and $\cap X_\alpha \neq \emptyset$ since each X_α contains x. Hence (***) and Problem 4 imply that A is connected, so by Theorem 12.2 A is an interval.

It remains to show that A is an open interval. Assume the contrary – then A has one of the forms $[a, b]$, $[a, b)$, $(a, b]$, $[a, +\infty)$, $(-\infty, b]$. We will treat the case $A = [a, b]$; other cases are analogous.

So assume that $A = [a, b]$. Then $b \in U$; since U is open, there is $\varepsilon > 0$ such that $(b - \varepsilon, b + \varepsilon) \subseteq U$. But then $[b, b + \frac{\varepsilon}{2}] \subseteq U$ as well, so by definition $b \sim b + \frac{\varepsilon}{2}$, so $b + \frac{\varepsilon}{2} \in A$, a contradiction.

(c) Since distinct equivalence classes are disjoint and the union of all equivalence classes is the entire set on which the equivalence relation is defined, (b) implies that U is a disjoint union of open intervals (call these intervals A_α), and we only need to show that the number of intervals is at most countable.

We know that every non-empty open interval contains a rational number. Choose one rational number q_α in each interval A_α. Since distinct A_α are disjoint, the chosen numbers q_α will all be distinct. Thus we have constructed an injective map from the set of intervals $\{A_\alpha\}$ to $\mathbb{Q}$ or, equivalently, a bijection from $\{A_\alpha\}$ to a subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is countable and a subset of a countable set is at most countable, we deduce that the number of intervals A_α is at most countable.

8. Use Problem 6 to show that the analogue of Problem 7 does not hold in $\mathbb{R}^2$, that is, there exist open subsets of $\mathbb{R}^2$ which are not representable as disjoint unions of open discs (an open disc is an open ball in $\mathbb{R}^2$).

Solution: Let Y be any non-empty connected open subset of $\mathbb{R}^2$ which is not an open disk (e.g. we can let Y be an open rectangle $(a, b) \times (c, d)$– it is connected by Problem 6). Suppose that Y is a disjoint union of open disks $\{U_\alpha\}$. There must be more than one disk in this collection, so if we set $A = U_\alpha$ (for some fixed α) and $B = \cup_{\beta \neq \alpha} U_\beta$, then $Y = A \sqcup B$, both A and B are non-empty and both A and B are open (since open disks are open and arbitrary unions of open sets are open). This contradicts the assumption that Y is connected.