Solutions to Homework #5

1. Let A be a non-empty bounded above subset of $\mathbb{R}$. Prove that $\sup(A) \in \overline{A}$ directly from the definition of a contact point and the definition of supremum. You should give a short clean argument.

Solution: Let $M = \sup(A)$. We prove that $M \in \overline{A}$ by contradiction. Suppose $M \notin \overline{A}$. Then M is not a contact point of A, so there exists $\varepsilon > 0$ such that $A \cap (M - \varepsilon, M + \varepsilon) = \emptyset$. In other words, the interval $(M - \varepsilon, M + \varepsilon)$ contains no elements of A. Note that the interval $[M + \varepsilon, \infty)$ also contains no elements of A (since all elements of A are $\leq M$). Hence A is contained in $\mathbb{R} \setminus ((M - \varepsilon, M + \varepsilon) \cup [M + \varepsilon, \infty)) = (-\infty, M - \varepsilon]$. But this means that A is bounded above by $M - \varepsilon$, so $\sup(A) \leq M - \varepsilon$, which is a contradiction.

2. Prove Lemma 10.6 from class: if $\{x_n\}$ is a Cauchy sequence in some metric space X, and $\{x_n\}$ contains a convergent (in X) subsequence, then $\{x_n\}$ converges in X. You may use the fact that every metric space X is a subset of some complete metric space Y (we will prove this in Lecture 13).

Solution: As suggested above, we assume that $X \subseteq Y$ where Y is complete. Since $\{x_n\}$ is a Cauchy sequence in X, it is also a Cauchy sequence in Y, and since Y is complete, $\{x_n\}$ converges to some $y \in Y$. Hence every subsequence of $\{x_n\}$ also converges to y. On the other hand, we are given that some subsequence $\{x_{n_k}\}$ converges to some $x \in X$. By uniqueness of the limit of a sequence, we must have $y = x$ and thus $y \in X$, so $\{x_n\}$ converges in X.

3. Let Z be a metric space and let Y be a dense subset of Z. Suppose that every Cauchy sequence in Y converges in Z. Prove that Z is complete.

Solution: Let $\{z_n\}$ be an arbitrary sequence in Z. Since Y is dense in Z, for each $n \in \mathbb{N}$ the intersection $N_{1/n}(z_n) \cap Y$ is non-empty, so we can find $y_n \in Y$ such that $d(y_n, z_n) < \frac{1}{n}$; in particular, $d(y_n, z_n) \to 0$ as $n \to \infty$.

Now assume that $\{z_n\}$ is Cauchy. We claim that in this case $\{y_n\}$ constructed above is also Cauchy. Indeed, fix $\varepsilon > 0$. Since $\{z_n\}$ is Cauchy, there exists $M_1 \in \mathbb{N}$ such that $d(z_n, z_m) < \frac{\varepsilon}{2}$ for all $n, m \geq N$. Choose $M_2 \in \mathbb{N}$ such that $\frac{1}{M_2} < \frac{\varepsilon}{4}$. Let $M = \max\{M_1, M_2\}$. We claim that $d(y_n, y_m) < \varepsilon$ for all $n, m \geq M$ (whence $\{y_n\}$ is Cauchy). Indeed, by quadrilateral inequality we have

$$d(y_n, y_m) \leq d(y_n, z_n) + d(z_n, z_m) + d(z_m, y_m) < \frac{1}{n} + \frac{\varepsilon}{2} + \frac{1}{m} \leq \frac{\varepsilon}{2} + \frac{2}{M_2} \leq \varepsilon.$$

Since $\{y_n\}$ is a Cauchy sequence in Y, by assumption it converges to some $z \in Z$, so $d(y_n, z) \to 0$ as $n \to \infty$. Since $0 \leq d(z_n, z) \leq d(z_n, y_n) + d(y_n, z)$ and $d(z_n, y_n) \to 0$ by construction, by the squeeze theorem we conclude that $d(z_n, z) \to 0$ as $n \to \infty$, so $\{z_n\}$ converges to z.

Thus, we proved that every Cauchy sequence in Z converges in Z, so by definition Z is complete.

4. Let X be a set, and let d_1 and d_2 be two different metrics on X. Given $x \in X$ and $\varepsilon > 0$, define $N^1_\varepsilon(x) = \{y \in X : d_1(y, x) < \varepsilon\}$, the ε-neighborhood of x with respect to d_1, and similarly define $N^2_\varepsilon(x) = \{y \in X : d_2(y, x) < \varepsilon\}$. We will say that d_1 and d_2 are *topologically equivalent* if a subset S of X is open with respect to $d_1 \iff$ it is open with respect to d_2. (**Note:** for brevity, if d is a metric on X, we will say that S is d-open if S is open as a subset of the metric space (X, d)).

(a) Prove that d_1 and d_2 are topologically equivalent if and only if for every $\varepsilon > 0$ and every $x \in X$ there exist $\delta_1, \delta_2 > 0$ (depending on both ε and x) such that $N^1_{\delta_1}(x) \subseteq N^2_\varepsilon(x)$ and $N^2_{\delta_2}(x) \subseteq N^1_\varepsilon(x)$.

(b) Suppose that there exist real numbers $A, B > 0$ such that $d_1(x, y) \leq Ad_2(x, y)$ and $d_2(x, y) \leq Bd_1(x, y)$ for all $x, y \in X$. Use (a) to prove that d_1 and d_2 are topologically equivalent.

(c) Now use (b) to prove that the Euclidean and Manhattan metrics on $\mathbb{R}^n$ are topologically equivalent.

Solution: To simplify the terminology further, we will say that a set S is d_1-open if S is open with respect to d_1 (and similarly for d_2).

(a) "$\Rightarrow$" Assume that d_1 and d_2 are topologically equivalent. Take any $\varepsilon > 0$ and $x \in X$. We know that the set $N^2_\varepsilon(x)$ is d_2-open, hence by assumption it is also d_1-open. By definition, the latter means that for any $y \in N^2_\varepsilon(x)$

there exists $\delta > 0$ such that $N^1_\delta(y) \subseteq N^2_\varepsilon(x)$. In particular, this is true for $y = x$, so there exists some $\delta_1 > 0$ such that $N^1_{\delta_1}(x) \subseteq N^2_\varepsilon(x)$. Similarly, there must exist $\delta_2 > 0$ such that $N^2_{\delta_2}(x) \subseteq N^1_\varepsilon(x)$.

"$\Leftarrow$" Now assume that for every $\varepsilon > 0$ and every $x \in X$ there exist $\delta_1, \delta_2 > 0$ (depending on both ε and x) such that $N^1_{\delta_1}(x) \subseteq N^2_\varepsilon(x)$ and $N^2_{\delta_2}(x) \subseteq N^1_\varepsilon(x)$.

Let U be any d_1-open set, and take any $x \in U$. Then there must exist $\varepsilon > 0$ such that $N^1_\varepsilon(x) \subseteq U$. By assumption, there exists $\delta_2 > 0$ such that $N^2_{\delta_2}(x) \subseteq N^1_\varepsilon(x)$, so $N^2_{\delta_2}(x) \subseteq U$. This proves that U is d_2-open. Similarly, we argue that any d_2-open set must be d_1-open, whence d_1 and d_2 are topologically equivalent.

(b) Take any $\varepsilon > 0$ and $x \in X$, and let $\delta_1 = \frac{\varepsilon}{B}$. We claim that $N^1_{\delta_1}(x) \subseteq N^2_\varepsilon(x)$. Indeed, if $y \in N^1_{\delta_1}(x)$, then $d_1(y,x) < \delta_1 = \frac{\varepsilon}{B}$, whence $d_2(y,x) \leq Bd_1(y,x) < \varepsilon$, so $y \in N^2_\varepsilon(x)$. Similarly, the inclusion $N^2_{\delta_2}(x) \subseteq N^1_\varepsilon(x)$ always holds for $\delta_2 = \frac{\varepsilon}{A}$. Thus, we obtained the desired inclusion between open balls with respect to d_1 and d_2, so by (a), d_1 and d_2 are topologically equivalent.

(c) Let us denote the Euclidean metric by d_E and the Manhattan metric by d_M. We claim that for any $x, y \in \mathbb{R}^n$ we have

$$d_E(x,y) \leq d_M(x,y) \leq \sqrt{n} \cdot d_E(x,y). \qquad (***)$$

By (b), this would imply that d_M and d_E are equivalent.

Recall that if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then $d_M(x,y) = \sum_{i=1}^n a_i$ and $d_E(x,y) = \sqrt{\sum_{i=1}^n a_i^2}$ where $a_i = |x_i - y_i|$. Thus, we need to show that for any non-negative real numbers $a_1, \ldots, a_n$, the following inequalities hold:

$$\sqrt{\sum_{i=1}^n a_i^2} \leq \sum_{i=1}^n a_i \leq \sqrt{n} \cdot \sqrt{\sum_{i=1}^n a_i^2}$$

The first inequality is easy: indeed, $(\sum_{i=1}^n a_i)^2 = \sum_{i=1}^n a_i^2 + 2\sum_{i<j} a_i a_j \geq \sum_{i=1}^n a_i^2$, and taking square roots of both sides, we get the result.

The second inequality can be proved by direction computation using the inequality $2ab \leq a^2 + b^2$ for all $a, b \in \mathbb{R}$ (which holds since $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$). A more conceptual way is to use Cauchy-Schwartz inequality (see Theorem 9 on page 23 in Pugh). Pugh states the inequality in the (most natural) vector form. The scalar form of the Cauchy-Schwartz inequality asserts that for any $x_1, \dots, x_n, y_1 \dots, y_n \in \mathbb{R}$ we have

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_k^2} \cdot \sqrt{\sum_{i=1}^{n} y_k^2}$$

Setting $x_i = 1$ and $y_i = a_i$ for all i, we get $\sum_{i=1}^{n} a_i \leq \sqrt{n} \cdot \sqrt{\sum_{i=1}^{n} a_i^2}$, exactly what we needed to prove.

5.

(a) Theorem 40 in Pugh states the following: Let X, Y are metric metric spaces, assume that X is a compact, and assume that $f : X \to Y$ is continuous and bijective. Then $f^{-1} : Y \to X$ is also continuous. Give a short proof of this theorem by combining Corollary 7.2, Theorem 8.4, Theorem 9.2 and Theorem 11.1 from class (the respective references in Pugh are Theorem 11, equivalence of (i) and (iii), Theorem 26, Theorem 32 and Theorem 36).

(b)* Use (a) to show that there exist metric spaces X and Y and a function $f : X \to Y$ such that f is continuous and bijective, but $f^{-1} : Y \to X$ is not continuous (so that the assumption that X is compact in (a) is essential).

Solution: (a) By Corollary 7.2 from class, to prove that $f^{-1} : Y \to X$ is continuous, it suffices to prove that the preimage of any closed subset C of X under f^{-1} is closed in Y; in other words, we need to show that if C is closed in X, then $(f^{-1})^{-1}(C)$ is closed in Y. Since f is bijective, we have $(f^{-1})^{-1}(C) = f(C)$, so we need to show that if C is closed in X, then $f(C)$ is closed in Y.

So, suppose that C is closed in X. Since X is compact, by Theorem 9.2, C is compact. Since f is continuous, by Theorem 11.1, $f(C)$ is compact. Finally, by Theorem 8.4, a compact set is closed in any metric space containing it, so $f(C)$ is closed in Y. This completes the proof.

(b) Let $X = \mathbb{R}$ with discrete metric and $Y = \mathbb{R}$ with usual metric. We claim that the identity function $\iota : X \to Y$ has the required properties.

First observe that any subset of X is open: indeed, points (1 element subsets) of X are open since $N_1(x) = \{x\}$ for all $x \in X$. And since any subset S of X is the union of its points and arbitrary unions of open sets are open, we conclude that any $S \subseteq X$ is open.

Next we claim that the identity function $\iota : X \to Y$ is continuous. This holds by Problem 5(b) in HW#3, but also follows immediately from Theorem 7.1 and the above observation that any subset of X is open.

It is clear that ι is bijective.

Suppose that the inverse map $\iota^{-1} : Y \to X$ is continuous. If S is any subset of $\mathbb{R}$, then S is open in X as observed above, so $\iota(S) = (\iota^{-1})^{-1}(S)$ is open in Y by Theorem 7.1. But $\iota(S) = S$ as sets. It follows that any subset of Y is open, which is clearly not the case (e.g. points in Y are not open).