Solutions to Homework #4

1. Let $f : A \to B$ be a function. Give a detailed proof of the following properties:

(a) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ for all $U, V \subseteq B$

(b) $f(f^{-1}(D)) \subseteq D$ for all $D \subseteq B$. Give an example showing that the inclusion may be strict.

(c) $f^{-1}(f(C)) \supseteq C$ for all $C \subseteq A$. Give an example showing that the inclusion may be strict.

Solution: (a) Since both sides of the equality are subsets of B, we need to show that given any $x \in B$, we have

$$x \in f^{-1}(U \cap V) \iff x \in f^{-1}(U) \cap f^{-1}(V).$$

So let $x \in B$. Then $x \in f^{-1}(U \cap V) \iff f(x) \in U \cap V \iff f(x) \in U$ and $f(x) \in V \iff x \in f^{-1}(U)$ and $x \in f^{-1}(V) \iff x \in f^{-1}(U) \cap f^{-1}(V)$.

(b) If K and L are sets, to prove $f(K) \subseteq L$ we need to show that $f(x) \in L$ for all $x \in K$. We need to check the latter condition in the case $L = D$ and $K = f^{-1}(D)$. But the assumption $x \in K = f^{-1}(D)$ means that $f(x) \in D = L$ just by definition of preimage, so we are done.

If $f : A \to B$ is any non-surjective function and $D = B$, then $f(f^{-1})(D) \subseteq f(A) \neq B = D$ (since f is not surjective).

(c) Let $c \in C$. Then $f(c) \in f(C)$, so $c \in \{x \in A : f(x) \in f(C)\}$, and the latter set is equal to $f^{-1}(f(C))$ (by definition of preimage).

Let $f : A \to B$ be any non-injective function, so there exist $a_1 \neq a_2$ in A with $f(a_1) = f(a_2)$. If we take $C = \{a_1\}$, then $f(C) = \{f(a_1)\}$ and $f^{-1}(f(C))$ contains at least two elements, a_1 and a_2, so in particular, $f^{-1}(f(C)) \neq C$.

2. Let $\{x_n\}$ be a sequence in a metric space (X, d), and let x be some element of X. Prove that the following conditions are equivalent:

(i) some subsequence of $\{x_n\}$ converges to x

(ii) for every $\varepsilon > 0$ there are infinitely many n for which $x_n \in N_\varepsilon(x)$.

When proving the implication (ii)⇒(i) make it clear how you use that $x_n \in N_\varepsilon(x)$ for infinitely many n (and not just for some n).

Solution: "(i)⇒ (ii)" Suppose that some subsequence $\{x_{n_k}\}$ converges to x. Then for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $x_{n_k} \in N_\varepsilon(x)$ for all $k \geq M$. In particular, $x_n \in N_\varepsilon(x)$ for infinitely many n (namely $n = n_M, n_{M+1}, \ldots$)

"(ii)⇒ (i)" Now assume that (ii) holds. Then we can construct a sequence of integers $n_1 < n_2 < \ldots$ s.t. $x_{n_k} \in N_{1/k}(x)$ for each k. The sequence is constructed inductively as follows – once $n_1, \ldots, n_{k-1}$ are chosen, using the fact that $x_n \in N_{1/k}(x)$ for infinitely many n, we can always choose one of those n's s.t. $n > n_{k-1}$; this n will be our n_k).

Since $x_{n_k} \in N_{1/k}(x)$, we have $d(x_{n_k}, x) < \frac{1}{k}$. Hence $d(x_{n_k}, x) \to 0$ as $k \to \infty$, and therefore $\{x_{n_k}\}$ converges to x.

3. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Prove that K is covering compact in two different ways:

(i) by showing that K is closed and bounded as a subset of $\mathbb{R}$ (We will prove in class next week that a subset of $\mathbb{R}$ is covering compact if and only if it is closed and bounded).

(ii) directly from definition of covering compactness.

Solution: (i) Clearly K is bounded since $K \subseteq [0,1] \subset (-1,1) = N_1(0)$. Also note that

$$\mathbb{R} \setminus K = (-\infty, 0) \cup (1, +\infty) \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n}).$$

Thus, $\mathbb{R} \setminus K$ is a union of open intervals, hence $\mathbb{R} \setminus K$ is open and therefore K is closed.

(ii) Let $\{U_\alpha\}$ be an open cover of K (where we consider K as a subset of $\mathbb{R}$). Since $0 \in K$, one of these sets, call it U_γ, must contain 0. Since U_γ is open, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq U_\gamma$. Now choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$ (so that $\frac{1}{N} < \varepsilon$). Then $(-\varepsilon, \varepsilon)$ contains 0 and $\frac{1}{n}$ for all $n \geq N$, so $K \subseteq U_\gamma \cup S$ where $S = \{1, \frac{1}{2}, \ldots, \frac{1}{N}\}$ (the important thing is that S is finite).

Now for each integer $1 \leq n \leq N$ we can find α_n such that $\frac{1}{n} \in U_{\alpha_n}$. Then $K \subseteq U_\gamma \cup \bigcup_{n=1}^{N} U_{\alpha_n}$, so we found a finite subcover of our original cover.

4. Let X be a metric space. Prove that X is covering compact $\iff X$ satisfies the following property:

(*) Let $\{K_\alpha\}$ be any collection of closed subsets of X such that for any finite subcollection $K_{\alpha_1}, \ldots, K_{\alpha_n}$, the intersection $K_{\alpha_1} \cap \ldots \cap K_{\alpha_n}$ is non-empty. Then the intersection of all sets in $\{K_\alpha\}$ is non-empty.

Solution: The above property (*) can be reformulated as follows (via contrapositive):

(**) Let $\{K_\alpha\}$ be any collection of closed subsets of X such that the intersection of all sets in $\{K_\alpha\}$ is empty. Then there is a finite subcollection $K_{\alpha_1}, \ldots, K_{\alpha_n}$ such that the intersection $K_{\alpha_1} \cap \ldots \cap K_{\alpha_n}$ is empty.

We will show that X is compact $\iff X$ satisfies (**).

"$\Rightarrow$" Suppose that X is compact, and let $\{K_\alpha\}$ be any collection of closed subsets of X such that the intersection of all sets in $\{K_\alpha\}$ is empty. Define $U_\alpha = X \setminus K_\alpha$. Then each U_α is open, and the fact that $\cap K_\alpha = \emptyset$ implies that $\cup U_\alpha = X$, so $\{U_\alpha\}$ is an open cover of X. Since X is compact, there exist $\alpha_1, \ldots, \alpha_n$ such that $X = \cup_{i=1}^n U_{\alpha_i}$. Then $\cap_{i=1}^n K_{\alpha_i} = \cap_{i=1}^n (X \setminus U_{\alpha_i}) = \emptyset$, as desired.

"$\Leftarrow$" Now assume that (**) holds. Take any open cover $\{U_\alpha\}$ of X, and define $K_\alpha = X \setminus U_\alpha$. Then each K_α is closed in X, and since $\cup U_\alpha = X$, we have $\cap K_\alpha = \emptyset$. By (**), we deduce that there exist $\alpha_1, \ldots, \alpha_n$ such that $\cap_{i=1}^n K_{\alpha_i} = \emptyset$. Then $X = \cup_{i=1}^n U_{\alpha_i}$, so $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{U_\alpha\}$. Thus we proved that X is compact.

5. Let $X = C[a, b]$, considered as a metric space with uniform metric d_{unif} (as defined in Problem 4 in HW#2). Prove that the set $B_1(\mathbf{0})$, the closed ball of radius 1 centered at $\mathbf{0}$ in X, is not sequentially compact. Here $\mathbf{0}$ is the function which is identically 0.

Solution: By Problem 7 in HW#2 there exists a sequence of functions $\{f_n\}$ in X such that $d_{unif}(f_n, f_m) = 1$ for all $n \neq m$. Moreover, an explicit construction of such sequence given in online solutions shows that one can choose $f_n \in B_1(\mathbf{0})$. We claim that the sequence $\{f_n\}$ has no convergent subsequences.

Indeed, suppose that $\{f_{n_k}\}$ is a convergent subsequence. Then $\{f_{n_k}\}$ is also Cauchy, so (by the definition of Cauchy with $\varepsilon = 1$), there exists $M \in \mathbb{R}$ such that $d_{unif}(f_{n_k}, f_{n_l}) < 1$ for all $k, l \geq M$. This is clearly a contradiction since $d_{unif}(f_{n_k}, f_{n_l}) = 1$ whenever $k \neq l$.

Recall that the notion of an *ultrametric* metric space was introduced in HW#3.7. The following problem gives an interesting example of an ultrametric metric space.

6. Let p be a fixed prime number. Define the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ as follows: given a nonzero $x \in \mathbb{Q}$, we can write $x = p^a \frac{c}{d}$ for some $a, c, d \in \mathbb{Z}$ where c and d are not divisible by p . Define $|x|_p = p^{-a}$ (note that the above representation is not unique, but it is easy to see that a is uniquely determined by x). For instance,

$$\left|\frac{9}{20}\right|_p = \begin{cases} \frac{1}{9} & \text{if } p = 3 \\ 4 & \text{if } p = 2 \\ 5 & \text{if } p = 5 \\ 1 & \text{for any other } p. \end{cases}$$

Also define $|0|_p = 0$. Now define the function $d_p : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}_{\geq 0}$ by $d_p(x, y) = |y - x|_p$.

(a) Prove that $(\mathbb{Q}, d_p)$ is an ultrametric space. (Note: the completion of this metric space is usually denoted by $\mathbb{Q}_p$ is called *p-adic numbers*).

(b) Describe explicitly the set $N_1(0)$ (the open ball of radius 1 centered at 0) in $(\mathbb{Q}, d_p)$.

(c) Let d be the standard metric on $\mathbb{Q}$ (that is, $d(x, y) = |y - x|$ where $|\cdot|$ is the usual absolute value). Give examples of sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{Q}$ such that

   (i) $x_n \to 0$ in $(\mathbb{Q}, d_p)$ but $\{x_n\}$ is unbounded as a sequence in $(\mathbb{Q}, d)$

   (ii) $y_n \to 0$ in $(\mathbb{Q}, d)$ but $\{y_n\}$ is unbounded as a sequence in $(\mathbb{Q}, d_p)$

Solution: (a) Take any $x, y, z \in \mathbb{Q}_p$, and let $u = y - x$, $v = z - y$, so that $u + v = z - x$. Thus, we need to prove that

$$|u + v|_p \leq \max\{|u|_p, |v|_p\} \text{ for all } u, v \in \mathbb{Q}_p.$$

The inequality is clear if $u = 0$, $v = 0$ or $u + v = 0$, so from now on we will assume that $u, v, u + v \neq 0$.

Write $u = p^a \cdot \frac{c}{d}$ and $v = p^b \cdot \frac{e}{f}$ where $p \nmid c, d, e, f$. By symmetry, without loss of generality we can assume that $a \geq b$. Then

$$u + v = p^b \cdot (p^{a-b}\frac{c}{d} + \frac{e}{f}) = p^b \cdot \frac{p^{a-b}cf + de}{df}.$$

Since $p \nmid d$ and $p \nmid f$, we have $p \nmid df$. We do not know whether $p^{a-b}cf + de$ is divisible by p, but in any case we can write $p^{a-b}cf + de = p^g \cdot h$ for some $g, h \in \mathbb{Z}$ with $g \geq 0$ and $p \nmid h$. Then $u + v = p^{b+g}\frac{h}{df}$, so

$$|u + v|_p = \frac{1}{p^{b+g}} \leq \frac{1}{p^b} = |v|_p \leq \max\{|u|_p, |v|_p\}.$$

(b) $N_1(0)$ consists of 0 and all rational numbers $\frac{m}{n}$ such that $gcd(m, n) = 1$ and $p \mid m$.

(c) The sequences $x_n = p^n$ and $y_n = \frac{1}{p^n}$ have the required properties.

7. Let X be an arbitrary metric space and $f : X \to \mathbb{R}$ a continuous function (where $\mathbb{R}$ is equipped with standard metric).

(i) Prove that the sets $\{x \in X : f(x) > 0\}$ and $\{x \in X : f(x) < 0\}$ are open and the set $\{x \in X : f(x) = 0\}$ is closed

(ii) Prove that if $g : X \to \mathbb{R}$ is another continuous function, then the set $\{x \in X : f(x) = g(x)\}$ is closed

Solution: (i) By definition $\{x \in X : f(x) > 0\} = f^{-1}((0, +\infty))$ (the preimage of $(0, +\infty)$ under f), $\{x \in X : f(x) < 0\} = f^{-1}((-\infty, 0))$ and $\{x \in X : f(x) = 0\} = f^{-1}(\{0\})$. Since $(0, +\infty)$ and $(-\infty, 0)$ are open subsets of $\mathbb{R}$ and $\{0\}$ is a closed subset of $\mathbb{R}$, the result follows from the fact that preimages of open (resp. closed) sets under continuous functions are open (resp. closed).

(ii) Note that $\{x \in X : f(x) = g(x)\} = \{x \in X : f(x) - g(x) = 0\}$. Since the function $f - g$ is continuous by Theorem 4.9 in Rudin, the result follows from (i).

Remark: The result of part (ii) remains true if g is a continuous function from X to Y where Y is an arbitrary metric space. However, the above argument does not work in this generality since addition or subtraction may not be defined in Y (and even if they are defined in some way, Theorem 4.9 from Rudin is not applicable).