Solutions to Homework #3.

1. Given a metric space (X, d), a point $x \in X$ and $\varepsilon > 0$, define $B_\varepsilon(x) = \{y \in X : d(y, x) \leq \varepsilon\}$, called the *closed ball of radius* ε *centered at* x.

(a) Prove that $B_\varepsilon(x)$ is always a closed subset of X.
(b) Deduce from (a) that $\overline{N_\varepsilon(x)} \subseteq B_\varepsilon(x)$, that is, the closure of the open ball of radius ε centered at x is contained in the respective closed ball.
(c) Is it always true that $\overline{N_\varepsilon(x)} = B_\varepsilon(x)$? Prove or give a counterexample.

Solution: (a) By definition, it suffices to show that $X \setminus B_\varepsilon(x)$ is open. Take any $z \in X \setminus B_\varepsilon(x)$. Then $d(z, x) > \varepsilon$, so if set $\delta = d(z, x) - \varepsilon$, then $\delta > 0$. Now take any $y \in N_\delta(z)$. Then $d(z, x) \leq d(y, x) + d(z, y) < d(y, x) + \delta$, whence $d(y, x) > d(z, x) - \delta = d(z, x) - (d(z, x) - \varepsilon) = \varepsilon$. Hence $y \in X \setminus B_\varepsilon(x)$.

Thus, we showed that for any $z \in X \setminus B_\varepsilon(x)$, there is $\delta > 0$ such that $N_\delta(z) \subseteq X \setminus B_\varepsilon(x)$, so by definition, $X \setminus B_\varepsilon(x)$ is open.

Another solution for (a): Here is a completely different solution to (a) which uses the fact that the preimage of a closed set under a continuous function is closed. Indeed, fix $x \in X$, and define the function $f : X \to \mathbb{R}$ by $f(y) = d(y, x)$. By Problem 6 in this homework, f is continuous. Note that $B_\varepsilon(x) = \{y \in X : f(y) \leq \varepsilon\} = f^{-1}((-\infty, \varepsilon])$. Since $(-\infty, \varepsilon]$ is a closed subset of $\mathbb{R}$ and f is continuous, the preimage $f^{-1}((-\infty, \varepsilon])$ is a closed subset of X, as desired.

(b) By Theorem 5.1(c) from class, any closed subset of X which contains $N_\varepsilon(x)$ must also contain $\overline{N_\varepsilon(x)}$. By (a), $B_\varepsilon(x)$ is a closed subset (which obviously contains $N_\varepsilon(x)$), so $\overline{N_\varepsilon(x)} \subseteq B_\varepsilon(x)$.

(c) No. For instance, take any set X with $|X| \geq 2$ and let d be the discrete metric on X ($d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$). Then for any $x \in X$ we have $N_1(x) = B_{1/2}(x) = \{x\}$ (one element set consisting of the point x itself). Thus, the set $\{x\}$ is both open and closed since it is simultaneously an open ball and a closed ball (in fact, it is easy to show that any subset of X is both open and closed). In particular, $\overline{N_1(x)} = N_1(x) = \{x\}$. On the other hand, the closed ball $B_1(x)$ is the entire space X, so $B_1(x) \neq N_1(x)$ since $|X| \geq 2$.

2. Let X be a metric space, and let $Z \subseteq Y$ be subsets of X. Prove that Z is closed as a subset of $Y \iff Z = Y \cap K$ for some closed subset K of X. Deduce that if Z is closed in X, then Z is closed in Y.

Solution: "$\Rightarrow$" Assume that Z is closed in Y. Then $Y \setminus Z$ is open in Y, so by the inheritance principle for open sets, there exists G open in X such that $Y \setminus Z = Y \cap G$. Then

$$Z = Y \setminus (Y \setminus Z) = Y \setminus (Y \cap G) = Y \setminus G = Y \cap (Y \setminus G) = Y \cap (X \setminus G),$$

and $X \setminus G$ is closed in X. Thus, if we set $K = X \setminus G$, then K is closed in X and $Y = K \cap X$.

"$\Leftarrow$" Assume that $Z = Y \cap K$ where K is closed in X. Then, by a similar computation $Y \setminus Z = Y \cap G$ where $G = X \setminus K$. Then G is open in X, so by the inheritance principle, $Y \setminus Z$ is open in Y and hence Z is closed in Y.

Finally, since $Z \subset Y$, we can always write $Z = Y \cap Z$, so if Z is closed in X, applying the above criterion, we deduce that Z is closed in Y.

3. Let (X, d) be a non-empty metric space and S a subset of X. Prove that the following three conditions are equivalent (as defined in class, S is called *bounded* if it satisfies either of those conditions):

(i) There exists $x \in X$ and $R \in \mathbb{R}$ such that $S \subseteq N_R(x)$.
(ii) For any $x \in X$ there exists $R \in \mathbb{R}$ such that $S \subseteq N_R(x)$.
(iii) The set $\{d(s,t) : s,t \in S\}$ is bounded above as a subset of $\mathbb{R}$.

Solution: "(ii)$\Rightarrow$ (i)" This is clear since $X \neq \emptyset$.

"(iii)$\Rightarrow$ (ii)" Let K be an upper bound for $\{d(s,t) : s,t \in S\}$. Take any $x \in X$. If $S = \emptyset$, there is nothing to prove. If $S \neq \emptyset$, fix some $s_0 \in S$ and let $R = K + 1 + d(s_0, x)$. Then for any $s \in S$ we have $d(s,x) \leq d(s, s_0) + d(s_0, x) \leq K + d(s_0, x) < R$, so $S \subseteq N_R(x)$.

"(i)$\Rightarrow$ (iii)" Suppose that $S \subseteq N_R(x)$ for some $x \in X$ and $M \in \mathbb{R}$. Then for any $s,t \in S$ we have $d(s,x) < R$ and $d(x,t) < R$, so by the triangle inequality $d(s,t) < 2R$. So the set $\{d(s,t) : s,t \in S\}$ is bounded above by $2R$.

Definition: Let (X, d) be a metric space and $\varepsilon > 0$. A subset S of X is called an *ε-net* if for any $x \in X$ there exists $s \in S$ such that $d(x,s) < \varepsilon$. In other words, S is an ε-net if X is the union of open balls of radius ε centered at elements of S.

4. Let S be a subset of a metric space (X, d). Prove that the following are equivalent:

(i) The closure of S is the entire X;
(ii) $U \cap S \neq \emptyset$ for any non-empty open subset U of X;

(iii) S is an ε-net for every $\varepsilon > 0$.

The subset S is called *dense* (in X) if it satisfies these equivalent conditions.

Solution: As suggested in the hint, we will prove that negations of (i), (ii) and (iii) are equivalent to each other. We start by explicitly formulating the negations, denoted below by (a), (b) and (c)

(a) The closure of S is strictly contained in X, that is, there exists $x \in X$ which is NOT a contact point of S

(b) There exists a non-empty open subset U of X such that $U \cap S = \emptyset$

(c) There exists $\varepsilon > 0$ such that S is not an ε-net.

Now we prove that (a), (b) and (c) are equivalent

"(a)$\Rightarrow$ (b)" If $x \in X$ is not a contact point of S, there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \cap S = \emptyset$. This means that (b) holds with $U = N_\varepsilon(x)$ (since $N_\varepsilon(x)$ is open and non-empty).

"(b)$\Rightarrow$ (a)" Take any $x \in U$ (such x exists since U is non-empty). Since U is also open, there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq U$. Since $U \cap S = \emptyset$, we must have $N_\varepsilon(x) \cap S = \emptyset$, so x is not a contact point of S.

"(a)$\Rightarrow$ (c)" As in implication "(a)$\Rightarrow$ (b)", there exists $x \in X$ and $\varepsilon > 0$ such that $N_\varepsilon(x) \cap S = \emptyset$. This means that $d(s, x) \geq \varepsilon$ for all $s \in S$, so S is not an ε-net.

"(c)$\Rightarrow$ (a)" Analogous to the proof of "(a)$\Rightarrow$ (c)".

5. Let X be any set with discrete metric ($d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$), and let Y be an arbitrary metric space.

(a) Let $\{x_n\}$ be a sequence in X. Prove that $\{x_n\}$ converges if and only if it is eventually constant, that is, there exists $M \in \mathbb{N}$ and $x \in X$ such that $x_n = x$ for all $n \geq M$.

(b) Prove that any function $f : X \to Y$ is continuous in two different ways: first using sequential definition of continuity and then using the ε-δ definition.

Solution: (a) "$\Leftarrow$" Suppose that there exists $M \in \mathbb{N}$ and $x \in X$ such that $x_n = x$ for all $n \geq M$. Then for any $\varepsilon > 0$ we have $d(x_n, x) = 0 < \varepsilon$ for all $n \geq M$, so $\{x_n\}$ converges to x. Of course, this direction holds in any metric space.

"$\Rightarrow$" Now suppose that $\{x_n\}$ converges to x. By the definition of limit with $\varepsilon = 1$, there exists $M \in \mathbb{N}$ such that $d(x_n, x) < 1$ for all $n \geq M$. But since the only possible values of d are 0 and 1, the inequality $d(x_n, x) < 1$ forces $d(x_n, x) = 0$ and hence $x_n = x$. Thus, $x_n = x$ for all $n \geq M$, as desired.

(b) Using the ε-δ definition. Fix an arbitrary $a \in X$. Given any $\varepsilon > 0$, choose $\delta = 1$. If $x \in X$ and $d(a, x) < 1$, we must have $x = a$ and hence $f(x) = f(a)$, so $d_Y(f(x), f(a)) = 0 < \varepsilon$. Thus, f is continuous at a, and since $a \in X$ was arbitrary, f is continuous on X.

Using the sequential definition. Again fix $a \in X$, and take any sequence $\{x_n\}$ in X which converges to a. By (a), there exists $M \in \mathbb{N}$ such that $x_n = a$ for all $n \geq M$. Then $f(x_n) = f(a)$ for all $n \geq M$ and (again by (a)), $\{f(x_n)\}$ converges to $f(a)$, whence f is continuous at a.

6.

(a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function such that

$$d_Y(f(u), f(v)) \leq d_X(u, v) \text{ for all } u, v \in X. \qquad (***)$$

Prove that f is continuous.

(b) Let (X, d) be a metric space, and fix $a \in X$. Use (a) to prove that the function $f : X \to \mathbb{R}$ (where $\mathbb{R}$ is equipped with the usual metric) given by $f(x) = d(a, x)$ is continuous. Warning: be careful with absolute values.

Solution: (a) Given any $\varepsilon > 0$, choose $\delta = \varepsilon$. If $u, v \in X$ with $d_X(u, v) < \delta$, then by (***) we have $d_Y(f(u), f(v)) \leq d_X(u, v) < \delta = \varepsilon$. Thus, by the ε-δ definition, f is continuous.

(b) In the notations of (a) we have $d_X = d$ and $d_Y(y_1, y_2) = |y_1 - y_2|$. By (a), to prove continuity of f it suffices to check that $|f(u) - f(v)| \leq d(u, v)$ for all $u, v \in X$ or equivalently,

$$|d(a, u) - d(a, v)| \leq d(u, v) \text{ for all } u, v \in X$$

Note that for any real number t the absolute value $|t|$ is equal to $\max\{t, -t\}$. Thus, if we want to prove $|t| \leq r$ where r is another real number, this is equivalent to showing that $t \leq r$ and $-t \leq r$.

By the triangle inequality, $d(a, u) \leq d(a, v) + d(v, u) = d(a, v) + d(u, v)$ and $d(a, v) \leq d(a, u) + d(u, v)$. Hence $d(a, u) - d(a, v) \leq d(u, v)$ and $-(d(a, u) - d(a, v)) = d(a, v) - d(a, u) \leq d(u, v)$. Therefore, using the above observation with $t = d(a, u) - d(a, v)$ and $r = d(u, v)$, we conclude that $|d(a, u) - d(a, v)| \leq d(u, v)$, as desired.

7. A metric space (X, d) is called **ultrametric** if for any $x, y, z \in X$ the following inequality holds:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

(Note that this inequality is much stronger than the triangle inequality). If X is any set and d is the discrete metric on X (that is, $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$), then clearly (X,d) is ultrametric. A more interesting example of an ultrametric space is given in the next problem.

Prove that properties (i) and (ii) below hold in any ultrametric space (X,d) (note that both properties are counter-intuitive since they are very far from being true in $\mathbb{R}$).

(i) Take any $x \in X$, $\varepsilon > 0$ and take any $y \in N_\varepsilon(x)$. Then $N_\varepsilon(y) = N_\varepsilon(x)$. This means that if we take an open ball of fixed radius around some point x, then for any other point y from that open ball, the open ball of the same radius, but now centered at y, coincides with the original ball. In other words, any point of an open ball happens to be its center.

(ii) Prove that a sequence $\{x_n\}$ in X is Cauchy $\iff$ for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $d(x_{n+1}, x_n) < \varepsilon$ for all $n \geq M$. **Note:** The forward implication holds in any metric space.

Solution: (i) Let $d = d(x,y)$. Since $y \in N_\varepsilon(x)$, we have $d < \varepsilon$. Take any $z \in N_\varepsilon(y)$. Then $d(y,z) < \varepsilon$, so $d(z,x) \leq \max\{d(y,z), d(x,y)\} = \max\{d(y,z), d\} < \varepsilon$, so $z \in N_\varepsilon(x)$. Thus we showed that $N_\varepsilon(y) \subseteq N_\varepsilon(x)$. The reverse inclusion $N_\varepsilon(x) \subseteq N_\varepsilon(y)$ is proved similarly.

(ii) The forward implication is clear. Assume now that for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $d(x_{n+1}, x_n) < \varepsilon$ for all $n \geq M$. Then for all $n \geq M$ we also have $d(x_{n+2}, x_{n+1}) < \varepsilon$ whence $d(x_{n+2}, x_n) \leq \max\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\} < \varepsilon$. Repeating the same trick several times, we deduce that $d(x_{n+p}, x_n) < \varepsilon$ for all $n \geq M$ and $p \geq 0$; equivalently, $d(x_m, x_n) < \varepsilon$ for all $n, m \geq M$. Therefore, the sequence $\{x_n\}$ is Cauchy.