Solutions to Homework #2

1. Let A be an uncountable set and B a countable subset of A.

(a) Prove that $A \setminus B$ is uncountable.
(b) Prove that A and $A \setminus B$ have the same cardinality.

Solution: (i) Suppose that $A \setminus B$ is countable. Then, $A = (A \setminus B) \cup B$ is a union of two countable sets, hence A is countable, contrary to our hypothesis.

(ii) As suggested in the hint, let C be a countably infinite subset of $A \setminus B$ (such C exists by Theorem 12 on page 33 in Pugh). Since B is countable and C is countably infinite, their union $B \cup C$ is also countably infinite, so there is a bijection $\phi : C \to B \cup C$. Now define the map $f : A \setminus B \to A$ by

$$f(x) = \begin{cases} x & \text{if } x \in (A \setminus B) \setminus C \\ \phi(x) & \text{if } x \in C. \end{cases}$$

It is clear that f is a bijection from $A \setminus B$ to A.

2. Let X and Y be any sets, and define X^Y to be the set of all functions $f : Y \to X$. Prove that if $|X| \geq 2$, then Y and X^Y do not have the same cardinality.

Solution: We prove this by contradiction. Assume that Y and X^Y have the same cardinality, that is, there exists a bijection $f : Y \to X^Y$. For each $y \in Y$ define $\alpha_y \in X$ by $\alpha_y = (f(y))(y)$, the value of the function $f(y) \in X^Y$ at the point y. Since $|X| \geq 2$, for each $y \in Y$ we can choose an element $\beta_y \in X$ such that $\beta_y \neq \alpha_y$. Now define the function $b \in X^Y$ by setting $b(y) = \beta_y$ for each $y \in Y$.

Since f is a bijection, there must exist $y \in Y$ such that $b = f(y)$. But then b and $f(y)$ have the same value at every point; in particular, $b(y) = (f(y))(y)$. The latter is impossible since $b(y) = \beta_y \neq \alpha_y = (f(y))(y)$.

3. Solution: (1) First we claim that if $A_1, \ldots, A_n$ is a finite collection of countable sets, their Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is countable. This was proved in class for $n = 2$, and the general case follows from the case $n = 2$ by induction (check details!)

(2) For each $n \in \mathbb{Z}_{\geq 0}$ let Z_n denote the set of polynomials with integer coefficients of degree at most n. Let $\mathbb{Z}^{n+1} = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{n+1 \text{ times}}$ be the Cartesian product of $n+1$ copies of $\mathbb{Z}$. Define the function $f : \mathbb{Z}^{n+1} \to Z_n$ by

$f((c_0,\dots,c_n)) = \sum_{k=0}^{n} c_k x^k$. Then f is surjective (by definition of Z_n); also, f is injective since two polynomials are equal if and only if they have the same coefficients in every degree. Thus, f is bijective. Since $\mathbb{Z}^{n+1}$ is countable by (1) above, we conclude that Z_n is also countable.

(3) By definition, the set of all polynomials with integer coefficients is equal to $\cup_{n=0}^{\infty} Z_n$. Since each Z_n is countable by (2), $\cup_{n=0}^{\infty} Z_n$ is a countable union of countable sets, hence $\cup_{n=0}^{\infty} Z_n$ is countable (by Lecture 3 or Corollary 18 on page 36 in Pugh).

(4) Finally, denote the set of all polynomials with integer coefficients by Z. For each polynomial $p \in Z$ let A_p be the set of its roots. Then by definition the set of all algebraic numbers is $\bigcup_{p\in Z} A_p$. Since each A_p is finite and Z is countable by (3), we can conclude that $\bigcup_{p\in Z} A_p$ is a countable union of finite sets, so it is countable.

4. Let $a \le b$ be real numbers and $X = C[a,b]$, the set of all continuous functions from $[a,b]$ to $\mathbb{R}$. Define the functions $d_{unif} : X \times X \to \mathbb{R}_{\ge 0}$ and $d_{int} : X \times X \to \mathbb{R}_{\ge 0}$ by

$$d_{unif}(f,g) = \max_{t\in[a,b]} |f(t)-g(t)| \text{ and } d_{int}(f,g) = \int_a^b |f(t)-g(t)|\,dt.$$

(a) Prove that (X, d_{unif}) is a metric space (the metric d_{unif} is called the **uniform metric**)

(b) (practice) Prove that (X, d_{int}) is a metric space (the metric d_{int} is called the **integral metric**)

Solution: (a) Since the absolute value of a real number cannot be negative, we always have $d_{unif}(f,g) \ge 0$. Also $d_{unif}(f,g) = 0 \iff \max_{t\in[a,b]} |f(t) - g(t)| = 0 \iff f(t)-g(t) = 0$ for all $t \in [a,b] \iff f = g$ as functions. Thus we verified (MS1)

(MS2) follows from the fact that $|-x| = |x|$ for all $x \in \mathbb{R}$. Indeed, $d_{unif}(f,g) = \max_{t\in[a,b]} |f(t)-g(t)| = \max_{t\in[a,b]} |-(f(t)-g(t))| = \max_{t\in[a,b]} |g(t)-f(t)| = d_{unif}(g,f)$.

It remains to prove (MS3), the triangle inequality. Take any $f,g,h \in X = C[a,b]$. Then for any $t \in [a,b]$ we have $|f(t)-h(t)| \le |f(t)-g(t)|+|g(t)-h(t)|$ by the triangle inequality in $\mathbb{R}$. Since

$$|f(t)-g(t)| \le \max_{s\in[a,b]} |f(s)-g(s)| = d_{unif}(f,g)$$

and similarly $|g(t) - h(t)| \leq d_{unif}(g,h)$, we conclude that $|f(t) - h(t)| \leq d_{unif}(f,g) + d_{unif}(g,h)$. Since this is true for any t, we have

$$d_{unif}(f,h) = \max_{t\in[a,b]} |f(t) - h(t)| \leq d_{unif}(f,g) + d_{unif}(g,h).$$

(b) As in (a), it is clear that $d_{int}(f,g) \geq 0$ for all f,g. But unlike (a), it is no longer clear that $d_{int}(f,g) > 0$ whenever $f \neq g$, so let us prove this. So assume $f,g \in C[a,b]$ with $f \neq g$, and let $u(t) = |f(t) - g(t)|$. Then u is a non-negative continuous function on $[a,b]$ and u is not identically zero (since $f \neq g$). Moreover, there exist $c \in (a,b)$ (open interval) such that $u(c) > 0$. By continuity there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq [a,b]$ and $u(t) > \frac{u(c)}{2}$ for all $t \in (c-\delta, c+\delta)$. We have

$$d_{int}(f,g) = \int_a^b u(t)\,dt = \int_a^{c-\delta} u(t)\,dt + \int_{c-\delta}^{c+\delta} u(t)\,dt + \int_{c+\delta}^{b} u(t)\,dt.$$

The first and third integrals are non-negative (since u is non-negative) and the second integral is bounded below by $\frac{u(c)}{2} \cdot ((c+\delta) - (c-\delta)) = u(c)\delta > 0$. Thus, $d_{int}(f,g) > 0$, as desired.

The second axiom ($d_{int}(f,g) = d_{int}(g,f)$) is proved similarly to (a). Finally, the triangle inequality follows by integrating both sides of the inequality $|f(t) - h(t)| \leq |f(t) - g(t)| + |g(t) - h(t)|$ from a to b (and using the property that if $u, v \in C[a,b]$ are such that $u(t) \leq v(t)$ for all $t \in [a,b]$, then $\int_a^b u(t)\,dt \leq \int_a^b v(t)\,dt$).

5. Let (X,d) be a metric space and S is a subset of X. Prove that S is open $\iff$ S is the union of some collection of open balls (which could be centered at different points).

Solution: The backwards direction is clear since (as proved in class) each open ball is an open set and the union of any collection of open sets is open. For the forward direction, assume that S is open. Then for every $x \in S$ there is $\varepsilon_x > 0$ such that $N_{\varepsilon_x}(x) \subseteq S$. We claim that $S = \cup_{x\in S} N_{\varepsilon_x}(x)$. Indeed, the union on the right-hand side is contained in S since each $N_{\varepsilon_x}(x)$ is contained in S. On the other hand, since $N_{\varepsilon_x}(x)$ always contains x, the union $\cup_{x\in S} N_{\varepsilon_x}(x)$ contains S, so we proved the reverse inclusion.

6. Let (X,d) be a metric space and S a subset of X.

(i) Recall from Lecture 4 that a point $x \in X$ is called a *contact point of* S if $N_\varepsilon(x) \cap S \neq \emptyset$ for all $\varepsilon > 0$

(ii) A point $x \in X$ is called an *interior point of* S if there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$

The set of all contact points of S is denoted by $\overline{S}$ and the set of all interior points of S is denoted by S^o.

(a) Prove that the set S^o is always open.

(b) Let $x \in S$. Prove that x is a contact point of $S \iff x$ is not an interior point of $X \setminus S$.

(c) Use (a) and (b) to prove that the set $\overline{S}$ is always closed (using the definition of a closed set given in class).

Solution: (a) Take any $x \in S^o$. By definition this means that there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$. We claim that $N_\varepsilon(x)$ is actually contained in S^o. Indeed, take any $y \in N_\varepsilon(x)$. Since $N_\varepsilon(x)$ is open, there exists $\delta > 0$ such that $N_\delta(y) \subseteq N_\varepsilon(x)$. But $N_\varepsilon(x) \subseteq S$, so $N_\delta(y) \subseteq S$, hence by definition $y \in S^o$.

Thus, we showed that every point in $N_\varepsilon(x)$ lies in S^o, so $N_\varepsilon(x) \subseteq S^o$, hence by definition S^o is open.

Note: The set S^o may very well be empty even if S is pretty big. For instance, if $X = \mathbb{R}$ and S is the set of all irrational numbers, then $S^o = \emptyset$.

(b) (x is a contact point of S) $\iff$ ($N_\varepsilon(x) \cap S \neq \emptyset$ for every $\varepsilon > 0$) $\iff$ ($N_\varepsilon(x) \not\subseteq X \setminus S$ for every $\varepsilon > 0$) $\iff$ (x is not an interior point of $X \setminus S$).

(c) By (a),$(X \setminus S)^o$ is open, hence its complement $X \setminus (X \setminus S)^o$ is closed. But $X \setminus (X \setminus S)^o$ is equal to $\overline{S}$ by (b), which finishes the proof.

7. Let $X = C[a, b]$ and $d = d_{unif}$ (as defined in Problem 4). Find an (infinite) sequence $f_1, f_2, \ldots$ of elements of X such that $d(f_i, f_j) = 1$ for all $i \neq j$.

Solution: Given real numbers r, s satisfying $a \leq r \leq s \leq b$, define $I_{r,s}$ to the unique function satisfying the following properties:

(i) $I_{r,s}(t) = 0$ for $t \in [a, r]$ and for $t \in [s, b]$

(ii) $I_{r,s}(\frac{r+s}{2}) = 1$ (note that $\frac{r+s}{2}$ is the midpoint of $[r, s]$)

(iii) $I_{r,s}$ is linear on each of the intervals $[r, \frac{r+s}{2}]$ and $[\frac{r+s}{2}, s]$.

(Draw the graph!) It is clear that each function $f_{r,s}$ is continuous (one can easily write down an explicit piecewise formula for $I_{r,s}$, but this is not necessary for this problem). It is also clear that if (r, s) and (r', s') are disjoint open intervals then the $d(I_{r,s}, I_{r',s'}) = 1$.

Thus, to solve the problem we just need to choose an infinite collection of pairwise disjoint open intervals (r_i, s_i) inside $[a, b]$ and let $f_i = I_{r_i,s_i}$. For instance, we can let $(r_i, s_i) = (a + \frac{b-a}{i+1}, a + \frac{b-a}{i})$ for each $i \in \mathbb{N}$.