Solutions to Homework #1 (except 1.4 and 1.6(b))

Problems:

1. Prove that $\mathbb{C}$ (complex numbers) cannot be made an ordered field (no matter how the order relation $<$ is defined). **Note:** in class we proved this for lexicographic order; the general proof is not much more complicated.

Solution: We argue by contradiction. Suppose $\mathbb{C}$ is an ordered field with respect to some order $<$. Since $i \neq 0$, by axiom (O1) we have $i > 0$ or $0 > i$.

Case 1: $i > 0$. By (OF2), we have $i^2 > 0$, that is, $-1 > 0$. Using (OF2) again, we have $(-1)^2 > 0$, that is, $1 > 0$. On the other hand, adding 1 to both sides of $-1 > 0$ (which we can by (OF1)), we get $0 > 1$. Thus, $1 > 0$ and $0 > 1$, which contradicts (O1).

Case 2: $0 > i$. By (OF1) we have $0 + (-i) > i + (-i)$, so $-i > 0$. By (OF2) we get $-1 = (-i)^2 > 0$, and we can proceed exactly as in Case 1 to reach a contradiction.

2. Let S be a nonempty subset of $\mathbb{R}$ bounded above, and let $-S = \{-x : x \in S\}$. Prove that $-S$ is bounded below and $\inf(-S) = -\sup(S)$.

Solution: Take any $y \in -S$. By definition $y = -x$ for some $x \in S$. Since $\sup(S)$ is an upper bound for S, we have $x \leq \sup(S)$ and hence $y = -x \geq -\sup(S)$. This proves that $-S$ is bounded below and $-\sup(S)$ is a lower bound for $-S$.

It remains to show that $-\sup(S)$ is the greatest lower bound for $-S$, that is, $z \leq -\sup(S)$ for any lower bound for $-S$. So let z be any lower bound for $-S$. Since $-S$ contains every element of the form $-x$ with $x \in S$, we have $z \leq -x$ for all $x \in S$ and hence $-z \geq x$ for all $x \in S$. Thus, $-z$ is an upper bound for S, so by definition of $\sup(S)$ we have $-z \geq \sup(S)$ and hence $z \leq -\sup(S)$, as desired.

3. Let S be an ordered set and A and B subsets of S such that

(i) $a \leq b$ for any $a \in A$ and $b \in B$;

(ii) $\sup(A)$ and $\inf(B)$ exist in S.

Prove that $\sup(A) \leq \inf(B)$.

Solution: Take any $b \in B$. Condition (i) implies that b is an upper bound for A and hence $\sup(A) \leq b$. Since this is true for every $b \in B$, we deduce that $\sup(A)$ is a lower bound for B and hence $\sup(A) \leq \inf(B)$.

5. Give a detailed and rigorous proof of the fact that

$$\lim_{n\to\infty} \frac{2n+3}{3n+4} = \frac{2}{3}$$

directly from the definition of limit of a sequence.

Solution: First note that

$$\left|\frac{2n+3}{3n+4} - \frac{2}{3}\right| = \left|\frac{1}{9n+12}\right| = \frac{1}{9n+12} < \frac{1}{n}. \qquad (***)$$

Now take any $\varepsilon > 0$. By the Archimedan property of $\mathbb{R}$, we can find $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. If we now take any natural number $n \geq N$, then by (***) we have

$$\left|\frac{2n+3}{3n+4} - \frac{2}{3}\right| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

which proves that $\lim\limits_{n\to\infty} \frac{2n+3}{3n+4} = \frac{2}{3}$ by the definition of limit.

6. Let $S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$. Prove by contradiction that S does not have a supremum in $\mathbb{Q}$ in two ways:

(a) Assuming the existence of $\mathbb{R}$ and the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, every non-empty open interval in $\mathbb{R}$ contains a rational number

(b) (bonus) using just $\mathbb{Q}$. For this problem use the description of S which does not involve $\sqrt{2}$, e.g. $S = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$.

Solution for (a): Suppose that $\sup(S)$ exists in $\mathbb{Q}$. We know that $\sqrt{2} \notin \mathbb{Q}$, so either $\sup(S) < \sqrt{2}$ or $\sup(S) > \sqrt{2}$.

First suppose $\sup(S) < \sqrt{2}$. By the density of $\mathbb{Q}$, the interval $(\sup(S), \sqrt{2})$ contains some $q \in \mathbb{Q}$. But then $q < \sqrt{2}$ (so $q \in S$) and $q > \sup(S)$, so $\sup(S)$ is not an upper bound for S, a contradiction.

Now suppose $\sup(S) > \sqrt{2}$. Then by density we can find $q \in \mathbb{Q}$ in the interval $(\sqrt{2}, \sup(S))$. Then $s < \sqrt{2} < q$ for all $s \in S$, so q is an upper bound for S and $q < \sup(S)$, so $\sup(S)$ is not the LEAST upper bound, again a contradiction.

Remark: There were several papers where the following argument was made: since $\sqrt{2}$ is an upper bound for S (by definition), if S does have a supremum in $\mathbb{Q}$, denoted $\sup(S)$, it must be true that $\sup(S) \leq \sqrt{2}$. However, this conclusion is not valid since $\sqrt{2} \notin \mathbb{Q}$, so the inequality $\sup(S) > \sqrt{2}$ would not contradict the definition of the least upper bound.

In fact, the following example shows that the inequality $\sup(S) > \sqrt{2}$ is actually possible. Indeed, let $T = S \cup \{2\}$, the set of all elements of S with the number 2 added. If we consider S as a subset of T, then S has the least upper bound in T, namely 2 (in fact, 2 is the only upper bound for S in T).

7. Deduce the Intermediate Value Theorem and Extreme Value Theorems directly from the following four results (which will be proved later in the course):

(1) Let $I = [a, b]$ be a closed bounded interval in $\mathbb{R}$, and consider I as a metric space with the standard metric ($d(x, y) = |x - y|$). Then I is compact and connected.

(2) Let $S \subseteq \mathbb{R}$ be a subset which is both compact and connected (again with respect to the standard metric). Then $S = \emptyset$ or $S = [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$.

(3) Let X and Y be metric spaces and $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected (as usual $f(X) = \{f(x) : x \in X\}$ is the image (=range) of f).

(4) Let X and Y be metric spaces and $f : X \to Y$ be a continuous function. If X is compact, then $f(X)$ is compact.

Solution: Let $f : I = [a, b] \to \mathbb{R}$ be a continuous function. By (1) I is compact and connected. By (3) and (4) $f(I)$ is compact and connected (and non-empty since it contains $f(a)$) and hence by (2) $f(I) = [m, M]$ for some $m, M \in \mathbb{R}$ with $m \leq M$.

Now we can prove both EVT and IVT. We start with EVT. Since $f(I) = [m, M]$ and $m, M \in [m, M]$, there exist $c, d \in I$ such that $f(c) = m$ and $f(d) = M$. Again since $f(I) = [m, M]$, for all $x \in I$ we have $m \leq f(x) \leq M$, that is, $f(c) \leq f(x) \leq f(d)$, which proves EVT.

Now we prove IVT. Take any $r \in \mathbb{R}$, which lies between $f(a)$ and $f(b)$. WOLOG assume $f(a) \leq f(b)$. Then $f(a) \leq r \leq f(b)$, $m \leq f(a) \leq M$ and $m \leq f(b) \leq M$, whence $m \leq f(a) \leq r \leq f(b) \leq M$. Thus, $r \in [m, M] = f(I)$, so there exists $e \in I$ such that $f(e) = r$. This proves IVT.