Solutions to Homework #11

Notation: Throughout the entire solutions the Lebesgue measure will be denoted by μ. This is done to avoid confusion with the standard notations m_k and M_k that naturally arise in Problems 2 and 4. In Problem 4 where we have to simultaneously work with the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^2$ we denote by μ_1 the Lebesgue measure on $\mathbb{R}$ and by μ_2 the Lebesgue measure on $\mathbb{R}^2$.

1. Let X be a metric space and $A \subseteq X$. Then

- A is a called an F_σ-set if A is countable union of closed sets.
- A is a called a G_δ-set if A a countable intersection of open sets.

(i) Prove that A is an F_σ-set $\iff$ $X \setminus A$ is a G_δ-set.

(ii) Prove that the collection of all F_σ-sets in X is closed under countable unions and the collection of all G_δ-sets in X is closed under countable intersections.

(iii)* Let $X = \mathbb{R}$ (with standard metric). Prove that every open subset of X is an F_σ-set and every closed subset of X is a G_δ-set.

Solution: (i) This follows directly from the de Morgan laws $(\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty} A_n^c$ and $(\cap_{n=1}^{\infty} A_n)^c = \cup_{n=1}^{\infty} A_n^c$ and the fact that complements of open sets are closed (and vice versa). Here $Y^c = X \setminus Y$, the complement of Y in X.

(ii) Suppose that B is a countable union of F_σ-sets, that is, $B = \cup_{n=1}^{\infty} B_n$ where each B_n is an F_σ-set, so $B_n = \cup_{n=1}^{\infty} B_{n,m}$ where each $B_{n,m}$ is closed. Note that we can write $B = \bigcup_{(n,m)\in\mathbb{N}\times\mathbb{N}} B_{n,m}$. Since $\mathbb{N} \times \mathbb{N}$ is countable, it follows that B is a countable union of closed sets, so by definition B is an F_σ-set. Thus, the collection of all F_σ-sets in X is closed under countable unions. Similarly, the collection of all G_δ-sets in X is closed under countable intersections.

(iii) First we show that every open interval in $\mathbb{R}$ is an F_σ-set. Indeed, if $I = (a, b)$ with $a, b \in \mathbb{R}$, we can write $I = \cup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$ (where we adopt the convention that $[c, d] = \emptyset$ if $c > d$). If $I = (a, +\infty)$, we write $I = \cup_{n=1}^{\infty} [a + \frac{1}{n}, +\infty)$ (all the intervals on the right-hand side are closed); for $I = (-\infty, b)$ we write $I = \cup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$, and finally $(-\infty, +\infty)$ is already closed.

By Problem 7 in HW#6, every open subset of $\mathbb{R}$ is at most countable union of open intervals. Since we already proved that open intervals are

F_σ-sets, the result follows from (ii) (here we use the fact that a finite union of F_σ-sets can also be written as a countable union of F_σ-sets since we can write $A = \cup_{n=1}^{\infty} A$).

2. Let $D : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function (defined by $D(x) = 1$ if $x \in \mathbb{Q}$ and 0 if $x \notin \mathbb{Q}$), and let $a < b$ be real numbers.

(a) Prove that D is Lebesgue-integrable on $[a, b]$ and that $\int\limits_{[a,b]} D\,dm = 0$.

(b) Prove that D is not Riemann-integrable on $[a, b]$.

Solution: (a) The image of D has two elements 0 and 1. The set $A_1 = \{x \in [a, b] : D(x) = 1\} = [a, b] \cap \mathbb{Q}$ is measurable and has measure 0 since it is countable, hence the set $A_0 = \{x \in [a, b] : D(x) = 0\} = [a, b] \setminus A_1$ is also measurable with $\mu(A_0) = \mu([a, b]) = \mu(A_1) = b - a$. Therefore, D is Lebesgue-integrable and $\int\limits_{[a,b]} D\,d\mu = 0{\cdot}\mu(A_0)+1{\cdot}\mu(A_1) = 0{\cdot}(b-a)+1{\cdot}0 = 0$.

(b) Let P be an arbitrary partition of $[a, b]$ into intervals $I_1, \ldots, I_n$ with lengths $\triangle x_1, \ldots, \triangle x_n$. Since each I_k contains both rational and irrational numbers, we have $M_k = 1$ and $m_k = 0$ where (in Pugh's notation from p. 166) $M_k = \sup\{D(x) : x \in I_k\}$ and $m_k = \inf\{D(x) : x \in I_k\}$. Hence the upper sum $U(D, P) = \sum_{k=1}^{n} M_k \triangle x_k = \sum_{k=1}^{n} \triangle x_k = b - a$ and the lower sum $L(D, P) = \sum_{k=1}^{n} m_k \triangle x_k = 0$. Thus, $\inf\{U(D, P)\}$, the infimum of upper sums over all partitions, is larger than $\sup\{L(D, P)\}$, the supremum of lower sums over all partitions, so by definition D is not Riemann/Darboux-integrable.

3. Let $\{f_n : [a, b] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of measurable functions, and let A be the set of all $x \in [a, b]$ such that $\{f_n(x)\}$ converges. Prove that A is measurable.

Solution: Following the hint, given $n, m, k \in \mathbb{N}$ let

$$A_{n,m,k} = \left\{ x \in [a, b] : |f_n(x) - f_m(x)| < \frac{1}{k} \right\}.$$

First we show that each $A_{n,m,k}$ is measurable. Indeed, if we fix n and m and let $g = f_{n,m}$, then

$$A_{n,m,k} = \left\{ \{x \in [a, b] : -\frac{1}{k} < g(x) < \frac{1}{k} \right\} = g_{<\frac{1}{k}} \cap g_{>-\frac{1}{k}}.$$

Since f_n and f_m are measurable, $g = f_n - f_m$ is also measurable by Lemma 24.2. By definition of a measurable function and Lemma 24.1 the sets $g_{<\frac{1}{k}}$ and

$g_{>-\frac{1}{k}}$ are both measurable, and finally their intersection is measurable by Lemma 22.3.

Once we know that each $A_{n,m,k}$ is measurable, Theorem 23.1 immediately implies that the set

$$\bigcap_{k=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n,m=N}^{\infty} A_{n,m,k} \qquad (*)$$

is measurable. It remains to show that the set given by (*) is equal to the set $\{x \in [a,b] : \{f_n(x)\} \text{ converges }\}$.

Indeed, take any $x \in [a,b]$. By the Cauchy criterion, the sequence $\{f_n(x)\}$ converges $\iff$ $\{f_n(x)\}$ is Cauchy $\iff$

$$\forall\, \varepsilon > 0 \;\; \exists\, N \in \mathbb{N} \text{ such that } |f_n(x) - f_m(x)| < \varepsilon \text{ for all } n, m \geq N. \qquad (**)$$

Since for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$, it is enough to require (**) for ε of the form $\frac{1}{k}$. In other words, $\{f_n(x)\}$ converges $\iff$ for every $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \frac{1}{k}$ for all $n, m \geq N$ or, equivalently,

$$\forall\, k \in \mathbb{N} \;\; \exists\, N \in \mathbb{N} \text{ such that } \forall\, n, m \geq N \text{ we have } x \in A_{n,m,k}\,. \qquad (***)$$

Finally, the definitions of unions and intersections immediately imply that x satisfies (***) $\iff$ x lies in the set given by (*), which completes the proof.

4. Given a function $f : [a,b] \to \mathbb{R}$, let $\Gamma(f)$ be the graph of f, that is,

$$\Gamma(f) = \{(x, f(x)) : x \in [a,b]\} \subset \mathbb{R}^2.$$

In each part of this problem prove that $\Gamma(f)$ has measure 0 (each part is a generalization of the previous part, but please do not deduce (a) from (b) or (b) from (c) as there are easier constructions that work for (a) and (b))

(a) $f(x) = x$

(b) f is an arbitrary continuous function

(c) f is an arbitrary measurable function

Solution: As mentioned at the beginning, in this problem we will denote the Lebesgue measure on $\mathbb{R}$ by μ_1 and the Lebesgue measure on $\mathbb{R}^2$ by μ_2. We will use analogous notations for outer measures.

In all 3 parts we will use the following criterion.

Claim 1: *Let $A \subseteq \mathbb{R}^2$ be a set, and suppose that for every $\varepsilon > 0$ there exists a countable cover $\{A_k\}$ of A such that $\sum\limits_k \mu_2^*(A_k) < \varepsilon$. Then $\mu_2^*(A) = 0$ (and hence A is measurable and has measure 0 by Observation 22.2).*

Remark: The sets A_k in the claim could be arbitrary; they do not even need to be measurable.

Proof of Claim 1: Let $\varepsilon > 0$. By hypotheses, there exists a countable collection $\{A_k\}$ such that $A \subseteq \bigcup_k A_k$ and $\sum_k \mu_2^*(A_k) < \varepsilon$. By (the 2-dimensional version of) Lemma 21.1(a)(c) we have $\mu_2^*(A) \leq \mu_2^*(\bigcup_k A_k) \leq \sum_k \mu_2^*(A_k)$, and so $\mu_2^*(A) < \varepsilon$. Since ε is arbitrary, it follows that $\mu_2^*(A) = 0$. $\square$

In each part we will prove that $\Gamma(f)$ has measure 0 using Claim 1.

(a) Let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $\frac{(b-a)^2}{n} < \varepsilon$. Let $A_1, \ldots, A_n$ be the squares with side length $\frac{b-a}{n}$ such that the left-lower vertex of A_1 is at the point (a, a) and for each $k \geq 2$, the left-lower vertex of A_k coincides with the right-upper vertex of A_{k-1} (thus, all A_k have their left-lower and right-upper vertices on $\Gamma(f)$, and the right-upper vertex of A_n is at (b, b)).

It is clear that $\Gamma(f) \subseteq \cup A_k$ and $\sum_{k=1}^{n} \mu_2^*(A_k) = n \cdot \frac{(b-a)^2}{n^2} = \frac{(b-a)^2}{n} < \varepsilon$, so we are done by Claim 1.

(b) Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x - y| < \delta$. Now choose $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta$. For each $0 \leq k \leq n$ let $x_k = a + k \cdot \frac{b-a}{n}$. For $1 \leq k \leq n$ let $M_k = \max\{f(x) : x \in [x_{k-1}, x_k]\}$ and $m_k = \min\{f(x) : x \in [x_{k-1}, x_k]\}$, and let A_k be the rectangle whose left-lower vertex is (x_{k-1}, m_k) and whose right-upper vertex is (x_k, M_k).

As in (a) the construction ensures that $\Gamma(f) \subseteq \cup A_k$. By the choice of δ we have $M_k - m_k \leq \frac{\varepsilon}{b-a}$, so $\mu_2^*(A_k) < \frac{b-a}{n} \frac{\varepsilon}{b-a} = \frac{\varepsilon}{n}$ and hence $\sum_{k=1}^{n} \mu_2^*(A_k) < \varepsilon$, and again we are done by Claim 1.

(c) Let $\varepsilon > 0$, and fix n such that $\frac{b-a}{n} < \varepsilon$.

For each $k \in \mathbb{Z}$ let $I_k = \{x \in [a, b] : \frac{k}{n} \leq f(x) < \frac{k+1}{n}\}$. Since f is measurable, each I_k is measurable. By construction $[a, b]$ is equal to the disjoint union $\sqcup_{k \in \mathbb{Z}} I_k$, so by countable additivity we have $\sum_{k \in \mathbb{Z}} \mu_1(I_k) = \mu_1([a, b]) = b - a$.

Now let $A_k = I_k \times [\frac{k}{n}, \frac{k+1}{n}]$. It is clear from the construction that $\Gamma(f) \subseteq \bigcup_{k \in \mathbb{Z}} A_k$.

By Theorem 23.2, each A_k is measurable (as a subset of $\mathbb{R}^2$) and $\mu_2(A_k) = \mu_1(I_k)\mu_1([\frac{k}{n}, \frac{k+1}{n}]) = \frac{\mu_1(I_k)}{n}$. Hence $\sum_{k \in \mathbb{Z}} \mu_2(A_k) = \frac{1}{n} \sum_{k \in \mathbb{Z}} \mu_1(I_k) = \frac{b-a}{n} < \varepsilon$, and we are done as in previous cases.

5. Let C be the standard Cantor set, and let $H : [0,1] \to [0,1]$ be the Cantor function AKA the Devil staircase function (see p.187 in Pugh).

(a) Prove that $\mu(H(C)) = 1$. This shows that a continuous function may send a set of measure zero to a set of positive measure.
(b) Compute $\int\limits_{[0,1]} H\, dm$.
(c) (bonus) Modify the construction of H to show that for every $\varepsilon > 0$ there exists a **strictly** increasing continuous function $f_\varepsilon : [0,1] \to [0,1]$ such that $\mu(H_\varepsilon(C)) > 1 - \varepsilon$.

Solution: (a) Since H is a non-decreasing continuous function with $H(0) = 0$ and $H(1) = 1$, by the intermediate value theorem we have $H([0,1]) = [0,1]$.

Next note that $H([0,1]) = H(C) \cup H([0,1] \setminus C)$, and by construction $H([0,1]\setminus C) \subset \mathbb{Q}$. Hence if we set $A = H([0,1]\setminus C)$, then $A \subseteq [0,1] \subseteq A \cup \mathbb{Q}$, and therefore $A\Delta[0,1] \subseteq \mathbb{Q}$.

Since $\mu^*(\mathbb{Q}) = 0$, we have $\mu^*(A\Delta[0,1]) = 0$. Since the set $[0,1]$ is elementary, A is measurable (by definition). Also, since $[0,1] \subseteq A \cup ([0,1]\Delta A)$, we have $1 = \mu([0,1]) \leq \mu(A) + \mu([0,1]\Delta A) = \mu(A)$; on the other hand, $A \subseteq [0,1]$, so $\mu(A) \leq \mu([0,1]) = 1$. Combining the two inequalities, we conclude that $\mu(A) = 1$, as desired.

(b) By Theorem 26.2(e) we have $\int\limits_{[0,1]} H\, d\mu = \int\limits_{C} H\, d\mu + \int\limits_{[0,1]\setminus C} H\, d\mu$, and since $\mu(C) = 0$ we have $\int\limits_{C} H\, d\mu = 0$ by Theorem 26.2(d).

For convenience denote by G the restriction of H to $[0,1] \setminus C$. It is clear that G is a simple function, so $\int\limits_{[0,1]\setminus C} H\, d\mu = \int\limits_{[0,1]\setminus C} G\, d\mu$ can be computed simply by computing the sum of a suitable series.

Note that $\mathrm{Im}\,(G)$ consists of all rational numbers in $(0,1)$ with denominator a power of 2. For each $c \in \mathrm{Im}\,(G)$ denote $A_c = \{x \in [0,1]\setminus C : G(x) = c\}$. By definition of G we have $\mu(A_{1/2}) = 1/3$, $\mu(A_{1/4}) = \mu(A_{3/4}) = 1/9$, $\mu(A_{1/8}) = \mu(A_{3/8}) = \mu(A_{5/8}) = \mu(A_{7/8}) = 1/27, \ldots,$ Therefore,

$$\int\limits_{[0,1]} g\, d\mu = \sum_{c\in \mathrm{Im}\,(g)} c\cdot\mu(A_c) = \tfrac{1}{3}\cdot\tfrac{1}{2} + \tfrac{1}{9}\cdot(\tfrac{1}{4}+\tfrac{3}{4}) + \tfrac{1}{27}\cdot(\tfrac{1}{8}+\tfrac{3}{8}+\tfrac{5}{8}+\tfrac{7}{8}) + \ldots$$

$$= \sum_{k=0}^{\infty}\frac{2^{k-1}}{3^{k+1}} = \frac{1}{6}\sum_{k=0}^{\infty}(\tfrac{2}{3})^k = \tfrac{1}{6}\cdot\tfrac{1}{1-2/3} = \tfrac{1}{2}.$$

6. (practice) Kolmogorov-Fomin, Problem 6 after Section 28 (p.292)

Solution: First we consider the special case where $f_n \to f$ everywhere (pointwise), not just almost everywhere. First we explain why the sets $E_k(\delta) = \{x : |f_k(x) - f(x)| \geq \delta\}$ are measurable.

Since each f_k is measurable and $f_k \to f$ pointwise, f is measurable by Lemma 24.3. Then $f_k - f$ is measurable (for any fixed k) by Lemma 24.2, whence $|f_k - f|$ is also measurable (see the argument in Problem 3). Finally, using Lemma 24.2 again we deduce that the sets $E_k(\delta)$ are measurable. Since countable unions and countable intersections of measurable sets are measurable, we deduce that the sets $R_n(\delta) = \cup_{k=n}^{\infty} E_k(\delta)$ and $M = \cap_{n=1}^{\infty} R_n(\delta)$ are also measurable.

Next we claim that $M = \emptyset$ (recall that so far we assume that $f_n \to f$ everywhere). Indeed, suppose that $x \in M$. Then $x \in R_n(\delta)$ for all $n \in \mathbb{N}$. Since $R_n(\delta) = \cup_{k=n}^{\infty} E_k(\delta)$, it follows that for any $n \in \mathbb{N}$ there exists $k > n$ such that $|f_k(x) - f(x)| \geq \delta$. This clearly implies that $f_k(x)$ does not converge to $f(x)$, contrary to our assumption.

Thus, $\cap_{n=1}^{\infty} R_n(\delta) = \emptyset$. Since clearly $R_1(\delta) \supseteq R_2(\delta) \supseteq \ldots$, by Theorem 11 in KF, Section 25, we deduce that $\mu(R_n(\delta)) \to \mu(\emptyset) = 0$. Since $E_n(\delta) \subseteq R_n(\delta)$, we have $\mu(E_n(\delta)) \leq \mu(R_n(\delta))$, so $\mu(E_n(\delta)) \to 0$ as $n \to \infty$ as well.

General case: We start with a lemma.

Lemma: *Let U be a measurable set and V any set such that $U \triangle V$ has measure 0. Then V is also measurable and $\mu(U) = \mu(V)$.*

Proof: Since $V \setminus U$ and $U \setminus V$ are both contained in $U \triangle V$ and $\mu(U \triangle V) = 0$, we must have $\mu^*(V \setminus U) = \mu^*(U \setminus V) = 0$, so by Observation 22.2(a) from class $V \setminus U$ and $U \setminus V$ are both measurable (and have measure zero). Then $U \cup V = U \sqcup (V \setminus U)$ is also measurable and $\mu(U \cap V) = \mu(U) + \mu(V \setminus U) = \mu(U) + 0 = \mu(U)$. Since $V = (U \cup V) \setminus (U \setminus V)$, we deduce that V is measurable; moreover, since $U \cup V = V \sqcup (U \setminus V)$, we have $\mu(V) = \mu(U \cup V) - \mu(U \setminus V) = \mu(U) - 0 = \mu(U)$. $\square$

We now go back to the problem. Let $A = \{x : f_n(x) \not\to f(x)\}$ (thus, by assumption $\mu(A) = 0$). Define the functions g_n and g by setting $g_n(x) = f_n(x)$, $g(x) = f(x)$ if $x \notin A$ and $g_n(x) = g(x) = 0$ if $x \in A$. Then by construction $g_n \to g$ everywhere.

We claim that each g_n is measurable. Indeed, if for fixed $n \in \mathbb{N}$ and $c \in \mathbb{R}$, we set $U = \{x : f_n(x) < c\}$ and $V = \{x : g_n(x) < c\}$, then $U \triangle V \subseteq A$. Since U is measurable (as f_n is measurable), by the above Lemma, V is also measurable, so g_n is measurable.

Thus, we can apply the result in the special case to the functions $\{g_n\}$ and g and deduce that the sets $E'_k(\delta) = \{x : |g_k(x) - g(x)| \geq \delta\}$ are measurable and $\mu(E'_k(\delta)) \to 0$ as $k \to \infty$. Finally, it is clear that $E'_k(\delta) \triangle E_k(\delta) \subseteq A$, so applying the above Lemma again, we deduce that $E_k(\delta)$ is measurable and $\mu(E_k(\delta)) = \mu(E'_k(\delta))$, so $\mu(E_k(\delta)) \to 0$ as $k \to \infty$.

7. Kolmogorov-Fomin, Problem 8 after Section 28 (p.292). **Note:** The functions $f_i^{(k)}$ are only defined for $1 \leq i \leq k$. It is probably useful to start by drawing the graphs of the first few functions in the sequence (say for $k = 1, 2, 3$).

Solution: Let $g_1, g_2, \ldots$ be the functions in the sequence (in the order in which they appear in the sequence), that is, $g_1 = f_1^{(1)}$, $g_2 = f_1^{(2)}$, $g_3 = f_2^{(2)}$, $g_4 = f_1^{(3)}$, $g_5 = f_2^{(3)}$, $g_6 = f_3^{(3)}$, $g_7 = f_1^{(4)}, \ldots$ In particular, $g_n = f_{i_n}^{(k_n)}$ where $k_n \to \infty$ as $n \to \infty$.

Let $\mathbf{0}$ denote the identically zero function. By definition, the function $f_i^{(k)}$ differs from $\mathbf{0}$ on a set of measure $\frac{1}{k}$, so $\mu\{x : |g_n(x) - \mathbf{0}(x)| \geq \delta\} \leq \frac{1}{k_n}$ for any $\delta > 0$. Since $k_n \to \infty$ as $n \to \infty$, it follows that $\mu\{x : |g_n(x) - \mathbf{0}(x)| \geq \delta\} \to 0$ as $n \to \infty$, so $g_n \to \mathbf{0}$ in measure.

On the other hand, for any $x \in (0, 1]$ and $k \geq 2$, there exist indices i and j such that $f_i^{(k)}(x) = 0$ and $f_j^{(k)}(x) = 1$. Thus, the sequence $\{g_n(x)\} = \{f_{i_n}^{(k_n)}(x)\}$ contains infinitely many 0's and infinitely many 1's, so it cannot be convergent.