## Solutions to Homework #10

**1.** Let $a < b$ be real numbers and let $\mathcal{P}_{even}[a,b] \subseteq C[a,b]$ be the set of all even polynomials (that is, polynomials which only involve even powers of $x$).

(a) Use Stone-Weierstrass Theorem to prove that $\mathcal{P}_{even}[a,b]$ is dense in $C[a,b] \iff 0 \notin (a,b)$.

(b)* (optional) Now prove the "$\Leftarrow$" direction in (a) using only Weierstrass Approximation Theorem (but not Stone-Weierstrass Theorem).

**Solution:** (a) Let $A = \mathcal{P}_{even}[a,b]$. Clearly, $A$ is an algebra. Since $1 \in \mathcal{P}_{even}[a,b]$, $A$ vanishes nowhere on $[a,b]$. If $0 \notin (a,b)$, then the function $f(x) = x^2$ (which lies in $A$) separates any two points on $[a,b]$, so by the Stone-Weierstrass Theorem, $A$ is dense in $C[a,b]$. On the other hand, if $0 \in (a,b)$, there exists $c \neq 0$ such that $[a,b]$ contains both $c$ and $-c$. Since $p(c) = p(-c)$ for all $p \in A$, we conclude that $A$ does not separate points and hence cannot be dense.

(b) Following the hint, we consider the case $0 \leq a < b$. Take any $g \in C[a,b]$. Note that the function $S : [a^2, b^2] \to [a,b]$ given by $S(x) = \sqrt{x}$ is continuous, so the composite function $h = f \circ S : [a^2, b^2] \to \mathbb{R}$ (in other words, $h(x) = f(\sqrt{x})$) is also continuous. By Weierstrass Approximation Theorem, for any $\varepsilon > 0$ there is a polynomial $p$ such that $|f(\sqrt{t}) - p(t)| < \varepsilon$ for all $t \in [a^2, b^2]$. Setting $x = \sqrt{t}$, we get that $|f(x) - p(x^2)| < \varepsilon$ for all $x \in [a,b]$. Since the function $x \mapsto p(x^2)$ lies in $\mathcal{P}_{even}[a,b]$, it follows that $\mathcal{P}_{even}[a,b]$ is dense in $C[a,b]$.

**2.**

(a)* Prove that the (direct) analogue of Weierstrass Approximation Theorem does not hold for $C(\mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$: Show that there exists $f \in C(\mathbb{R})$ which cannot be uniformly approximated by polynomials, that is, there is no sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $\mathbb{R}$.

(b)* Now prove that the following (weak) version of Weierstrass Approximation Theorem holds for $C(\mathbb{R})$: for any $f \in C(\mathbb{R})$ there exists a sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $[a,b]$ for any closed interval $[a,b]$ (of course, the point is that a single sequence will work for all intervals).

**Solution:** (a) Let $f : \mathbb{R} \to \mathbb{R}$ by any continuous bounded nonconstant function (e.g. $f(x) = \frac{1}{x^2+1}$). We claim that there is no sequence of polynomials $\{p_n\}$ such that $p_n \rightrightarrows f$ on $\mathbb{R}$.

Suppose, on the contrary, that such a sequence $\{p_n\}$ exists. Then there is $N \in \mathbb{N}$ such that $|f(x) - p_n(x)| < 1$ for all $n \geq N$ and $x \in \mathbb{R}$. Then $f(x) - 1 \leq p_n(x) \leq f(x) + 1$ for all $n \geq N$, $x \in \mathbb{R}$. Since $f$ is bounded, it follows that $p_n$ is bounded for all $n \geq N$, hence (since $p_n$ is a polynomial), $p_n$ is constant for all $n \geq N$.

Thus, $f$ is a limit of the sequence of constant functions $p_N, p_{N+1}, \ldots$ (since removing finitely many terms from a sequence does not affect its limit). But then it is clear that $f$ itself is constant, contrary to our assumption.

(b) The solution below does not really follow the given hint.

Since any closed interval $[a, b]$ is contained in $[-k, k]$ for some $k \in \mathbb{N}$ and uniform convergence on a set implies uniform convergence on any of its subsets, it suffices to find a sequence of polynomials $p_n$ such that $p_n \rightrightarrows f$ on $[-k, k]$ for each $k$.

For each $k \in \mathbb{N}$ we shall apply the Weierstrass Approximation Theorem on the interval $[-k, k]$ in the form "for every $\varepsilon > 0 \ldots$ " with $\varepsilon = \frac{1}{k}$. We conclude that there is a polynomial $p_k$ such that $d_{unif,k}(p_k, f) < \frac{1}{k}$ where $d_{unif,k}$ is the uniform metric on $C[-k, k]$; in other words, $|f(x) - p_k(x)| < \frac{1}{k}$ for all $x \in [-k, k]$.

Now fix $k \in \mathbb{N}$ and take any $n \geq k$. Then $[-k, k] \subseteq [-n, n]$, so for any $x \in [-k, k]$ we have $|p_n(x) - f(x)| < \frac{1}{n} \to 0$ uniformly in $x$. Therefore, $p_n \rightrightarrows f$ on $[-k, k]$.

**3.** Let $A_1, A_2, B_1$ and $B_2$ be subsets of the same set. Prove that

(a) $(A_1 \cup A_2) \triangle (B_1 \cup B_2) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$
(b) $(A_1 \cap A_2) \triangle (B_1 \cap B_2) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$

**Solution:** (a) Let $X = A_1 \cup A_2 \cup B_1 \cup B_2$. Since $U \triangle V = (U \cup V) \setminus (U \cap V)$, we have

$$(A_1 \cup A_2) \triangle (B_1 \cup B_2) = X \setminus ((A_1 \cup A_2) \cap (B_1 \cup B_2)) \subseteq X \setminus ((A_1 \cap B_1) \cup (A_2 \cap B_2))$$
$$\subseteq ((A_1 \cup B_1) \setminus (A_1 \cap B_1)) \cup ((A_2 \cup B_2) \setminus (A_2 \cap B_2)) = (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$$

(b) Let $X$ be as in (a), and let $Y^c = X \setminus Y$ for every subset $Y$ of $X$. For any $Y, Z$ we have

$$(Y^c \triangle Z^c) = (Y^c \cap (Z^c)^c) \cup ((Y^c)^c \cap Z^c) = (Y^c \cap Z) \cup (Y \cap Z^c) = Y \triangle Z.$$

Hence $(A_1 \cap A_2) \triangle (B_1 \cap B_2) = (A_1 \cap A_2)^c \triangle (B_1 \cap B_2)^c = (A_1^c \cup A_2^c) \triangle (B_1^c \cup B_2^c)$.

By (a), $(A_1^c \cup A_2^c) \triangle (B_1^c \cup B_2^c) \subseteq (A_1^c \triangle B_1^c) \cup (A_2^c \triangle B_2)^c = (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$. Combining the two inclusions, we obtain the assertion of (b).

**4.** In all parts of this problem $X = \mathbb{R}$ or $\mathbb{R}^2$, and $m$ denotes the Lebesgue measure on $X$.

(a)* Prove that every open subset of $X$ is measurable. Deduce that every closed subset of $X$ is measurable.

Now let $\Omega_0, \Omega_1, \Omega_2, \ldots$ be the following collections of subsets of $X$. First define $\Omega_0$ to be the set of all subsets of $X$ which are either open or closed. For each $k \geq 1$ define $\Omega_k$ to be the set of all subsets which can be represented either as a countable union or a countable intersection of subsets from $\Omega_{k-1}$.

(b) Deduce from (a) that each set in each $\Omega_k$ is measurable

(c) Assume that $X = \mathbb{R}$ and $S = \mathbb{Q}$. Does there exist $k \in \mathbb{N}$ such that $S \in \Omega_k$? If yes, what is the smallest such $k$?

(d) Same question as (c) for $S = \mathbb{R} \setminus \mathbb{Q}$

**Solution:** (a) We will give a proof in $\mathbb{R}^2$. The case of $\mathbb{R}$ is similar (and slightly easier). So let $A \subseteq \mathbb{R}^2$ be an open subset. We claim that $A$ is a countable union of rectangles (one can require that those rectangles are squares, but this is not necessary for this problem). Since rectangles are elementary (hence measurable) and countable unions of measurable sets are measurable (KF, Theorem 9 on p.264), this would imply that $A$ is measurable.

Take any point $P \in A$. Since $A$ is open, there exists $\varepsilon = \varepsilon(P) > 0$ such that $N_\varepsilon(P)$, the open disk of radius $\varepsilon$ centered at $P$ (in the Euclidean metric), is contained in $A$. Let $\delta = \frac{\varepsilon}{\sqrt{2}}$. Clearly, if $P = (a, b)$, then the open square $(a - \delta, a + \delta) \times (b - \delta, b + \delta)$ is contained in $N_\varepsilon(P)$ and hence also in $A$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find rational numbers $q_1, q_2, q_3, q_4$ such that $a - \delta < q_1 < a < q_2 < a + \delta$ and $b - \delta < q_3 < b < q_4 < b + \delta$, and let $R_P$ be the open rectangle $(q_1, q_2) \times (q_3, q_4)$.

We claim that $A = \cup_{P \in A} R_P$. Indeed, by construction each $R_P$ is contained in $A$, so $\cup_{P \in A} R_P \subseteq A$. On the other hand, since $P \in R_P$, we have $\cup_{P \in A} R_P \supseteq A$. Note that each $R_P$ has rational coordinates (of its vertices), and clearly there are countably many of rectangles with rational coordinates (there is a natural bijection between the set of such rectangles and a subset of $\mathbb{Q}^4 = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is countable being a Cartesian product of finitely many countable sets).

Thus, the collection $\{R_P : P \in A\}$ contains only countably many distinct subsets, so $A$ is a countable union of open rectangles.

We showed that every open subset of $\mathbb{R}^2$ is measurable. Since complements of measurable sets are measurable and since any closed set is the complement of an open set, we deduce that every closed set is measurable as well.

(b) This follows immediately from (the 2-dimensional version of) Theorem 23.1.

(c) The answer is yes, and the minimal $k$ is 1. Indeed, since $\mathbb{Q}$ is neither open nor closed, it does not lie in $\Omega_0$. On the other hand, $\mathbb{Q}$ is countable, so it can be written as a countable union of points. Since points are closed, we deduce that $\mathbb{Q} \in \Omega_1$.

(d) The answer is the same as in (c). For the same reason as in (c), $\mathbb{R}\backslash\mathbb{Q} \notin \Omega_0$. Now let $q_1, q_2, \ldots$ be any enumeration of $\mathbb{Q}$, and let $A_i = \mathbb{R}\backslash\{q_i\}$. Then each $A_i$ is open and $\mathbb{R} \setminus \mathbb{Q} = \bigcap\limits_{i=1}^{\infty} A_i$, so $\mathbb{R} \setminus \mathbb{Q} \in \Omega_1$.

**Remark:** In fact, it is not hard to see that each $\Omega_k$ is closed under taking complements. This is clear for $k = 0$, and for $k > 0$ this can be proved by induction using de Morgan's laws. Once we know that $\Omega_1$ is closed under complements, it follows immediately that the answers to (c) and (d) must be the same.

**5.**

(a) Let $A$ be a countable subset of $\mathbb{R}$. Prove that $A$ has measure zero (that is, $A$ is measurable and $m(A) = 0$).
(b) Prove that the (standard) Cantor set $C$ has measure 0 (see p.105 in Pugh for the definition of the standard Cantor set).

**Solution:** (a) First note that any point (that is, one element set) has measure zero since we can think of $\{a\}$ as a closed interval $[a, a]$. If $A$ is countable, it is a countable disjoint union of its points ($A = \sqcup_{a \in A}\{a\}$), so by countable additivity of $m$ (Theorem 23.1(d)) we have $m(A) = \sum\limits_{a \in A} m(\{a\}) = 0$.

(b) We present two solutions.

For the first solution, observe that proving that $C$ has measure 0 is equivalent to proving that its complement $[0,1]\backslash C$ is measurable and has measure 1. Since by construction, $[0, 1] \setminus C$ is a (countable) disjoint union of intervals, it is measurable, and its measure is equal to the sum of lengths of those intervals (again by Theorem 23.1). By construction, $[0,1]\setminus C$ is a disjoint union of 1 interval of length $\frac{1}{3}$, 2 intervals of length $\frac{2}{9}$, 4 intervals of length $\frac{4}{27}$ etc. (for each $n\in\mathbb{Z}_{\geq 0}$ we have $2^n$ intervals of length $\frac{1}{3^{n+1}}$). Therefore,

$$m([0,1]\setminus C)=\sum_{n=0}^{\infty}\frac{2^n}{3^{n+1}}=\frac{1}{3}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n=\frac{1}{3}\cdot\frac{1}{1-2/3}=1.$$

For the second solution, define the sets $C^0=[0,1]\supset C^1\supset C^2\supset\dots$ as on page 41 of Rudin (so that $C=\cap C^n$). By construction, each $C^n$ is a finite union of intervals of the same length, and $C^{n+1}$ is obtained from $C^n$ by removing the middle third of each interval of $C^n$. Thus, each $C^n$ is measurable and $m(C^{n+1})=\frac{2}{3}m(C^n)$. Since $m(C^0)=1$, we conclude that $m(C^n)=(\frac{2}{3})^n$.

Since $C\subset C^n$ for each $n$, we have $m^*(C)\leq m^*(C^n)=m(C^n)=(\frac{2}{3})^n$ for each $n$. Since $(\frac{2}{3})^n\to 0$ as $n\to\infty$, we conclude that $m^*(C)=0$. Hence by Observation 22.2(a), $C$ is measurable and $m(C)=0$.

**6.**

(a) Let $A,B$ and $C$ be subsets of the same set. Prove that

$$A\triangle C\subseteq(A\triangle B)\cup(B\triangle C)$$

(b) Now let $X=[0,1]$ or $[0,1]^2$. Let $A$ be a subset of $X$, and suppose that for every $\varepsilon>0$ there exists a measurable subset $B\subseteq X$ such that $m^*(A\triangle B)<\varepsilon$. Prove that $A$ is measurable.

**Solution:** (a) Let $x\in A\triangle C$. Then ($x\in A$ and $x\notin C$) or ($x\in C$ and $x\notin A$). WOLOG assume we are in the first case: $x\in A$ and $x\notin C$. If $x\in B$, then $x\in B\triangle C$, and if $x\notin B$, then $x\in A\triangle B$. In either case, we have $x\in(A\triangle B)\cup(B\triangle C)$.

(b) Take any $\varepsilon>0$. By assumption there exists a measurable set $B\subseteq X$ such that $m^*(A\triangle B)<\frac{\varepsilon}{2}$. Since $B$ is measurable, there exists an elementary set $C$ such that $m^*(B\triangle C)<\frac{\varepsilon}{2}$.

By Lemma 21.1(c), for any two sets $U$ and $V$ we have $m^*(U\cup V)\leq m^*(U)+m^*(V)$. Using this inequality and (a), we get $m^*(A\triangle C)\leq m^*(A\triangle B)+m^*(B\triangle C)<2\cdot\frac{\varepsilon}{2}=\varepsilon$. Since $\varepsilon>0$ was arbitrary, $A$ is measurable by definition.

**7.** Problem 7 from Kolmogorov-Fomin (p. 268). Note that the hint given in KF is essentially a sketch of the solutions. The things you need to justify are

(a) $C = \cup_{n=-\infty}^{\infty} \Phi_n$
(b) $\Phi_n \cap \Phi_m = \emptyset$ if $n \neq m$
(c) Assume that $\Phi_0$ is measurable. Then each $\Phi_n$ is measurable and $\mu(\Phi_n) = \mu(\Phi_0)$ for all $n \in \mathbb{Z}$
(d) the conclusion of (c) contradicts (33) in KF.

**Remark:** The Lebesgue measure on the circle $C$ can be defined in exactly the same way as on $\mathbb{R}$ with the exception that we call a subset of $C$ elementary if it is a finite union of arcs.

**Solution:** We start by introducing some additional notations and slightly rephrasing the definition of the sets $\Phi_n$.

Let $R : C \to C$ be the counterclockwise rotation by the angle $\pi\alpha$ (where $\alpha$ is a fixed irrational number). Define the relation $\sim$ on $C$ by $x \sim y \iff$ there exists $n \in \mathbb{Z}$ such that $y = R^n(x)$ (where $R^n = R$ applied $n$ times $=$ counterclockwise rotation by $n\pi\alpha$). It is straightforward to check that $\sim$ is an equivalence relation and that every equivalence class with respect to $\sim$ is countably infinite (the latter holds since $\alpha$ is irrational; if $\alpha$ was rational, equivalence classes would be finite).

Now let $\Phi_0$ be any subset of $C$ which contains precisely one element from each equivalence class, and for each $n \in \mathbb{Z}$ define $\Phi_n = R^n\Phi_0 = \{R^n(x) : x \in \Phi_0\}$. We now proceed with verifying properties (a)-(d) above.

(a) By construction each $\Phi_n$ lies in $C$, so $\cup_{n=-\infty}^{\infty} \Phi_n \subseteq C$. To prove the reverse inclusion, take any $x \in C$. By definition of $\Phi_0$ there exists $y \in \Phi_0$ such that $y \sim x$, so $x = R^n(y)$ for some $n \in \mathbb{Z}$. But then $x \in R^n\Phi_0 = \Phi_n$ as desired.

(b) Suppose, by way of contradiction, that there exists some $x \in C$ which lies in $\Phi_n \cap \Phi_m$ for some $n \neq m$. Then there exist $y, z \in \Phi_0$ such that $x = R^n(y) = R^m(z)$. Hence $y = R^{n-m}(z)$, so $y \sim z$. Since both $y$ and $z$ lie in $\Phi_0$ and since $\Phi_0$ contains only one element from each equivalence class, we must have $y = z$. Hence $y = R^{n-m}(y)$, which means that $R^{n-m}$ must be the trivial rotation. Since $R^{n-m}$ is the rotation by the angle $\pi\alpha(n-m)$, it follows that $\alpha(n-m) \in \mathbb{Z}$, which contradicts the assumption that $\alpha$ is irrational.

Next we prove a general lemma:

**Lemma:** *Let $f : C \to C$ be a rotation and $A$ any subset of $C$. Then*

(i) $\mu^*(f(A)) = \mu^*(A)$

(ii) *If $A$ is measurable, then $f(A)$ is also measurable and therefore $\mu(f(A)) = \mu(A)$ by (a)*

*Proof of Lemma:* (i) By definition,

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} length(I_n) \text{ where } \{I_n\}_{n=1}^{\infty} \text{ is a countable cover of } A \text{ by open arcs}\right\}.$$

Note that if $\{I_n\}_{n=1}^{\infty}$ is any such cover of $A$, then $\{f(I_n)\}_{n=1}^{\infty}$ is a countable cover of $f(A)$ by open arcs. Moreover $length(f(I_n)) = length(I_n)$ since $f$ is a rotation, so $\sum_{n=1}^{\infty} length(f(I_n)) = \sum_{n=1}^{\infty} length(I_n)$, and it follows that $\mu^*(f(A)) \leq \mu^*(A)$.

Since the last inequality holds for any subset $A$ and any rotation $f$ and since $f^{-1}$ is also a rotation, it remains true if we replace $A$ by $f(A)$ and $f$ by $f^{-1}$. This way we get $\mu^*(f^{-1}(f(A))) \leq \mu^*(f(A))$, that is, $\mu^*(A) \leq \mu^*(f(A))$.

Thus, $\mu^*(f(A)) \leq \mu^*(A)$ and $\mu^*(A) \leq \mu^*(f(A))$, so $\mu^*(A) = \mu^*(f(A))$.

(ii) Let $\varepsilon > 0$. Since $A$ is measurable, there exists an elementary set $B$ such that $\mu^*(A \triangle B) < \varepsilon$. By (i) we have $\mu^*(f(A \triangle B)) < \varepsilon$. Since $f$ is a bijection, it is clear that $f(A \triangle B) = f(A) \triangle f(B)$, so $\mu^*(f(A) \triangle f(B)) < \varepsilon$. Finally note that since $B$ is a finite disjoint union of arcs, $f(B)$ is also a finite disjoint union of arcs, so $f(B)$ is elementary and hence the previous inequality implies that $f(A)$ is measurable. □

(c) and (d) Assume that $\Phi_0$ is measurable. Then by the Lemma each $\Phi_n$ is measurable and $\mu(\Phi_n) = \mu(\Phi_0)$ for all $n$.

Since $C = \sqcup_{n \in \mathbb{Z}} \Phi_n$, by countable additivity of $\mu$ we have

$$1 = \mu(C) = \sum_{n=-\infty}^{n=\infty} \mu(\Phi_n) = \sum_{n=-\infty}^{n=\infty} \mu(\Phi_0).$$

This is impossible since if $\mu(\Phi_0) = 0$, then $\sum_{n=-\infty}^{n=\infty} \mu(\Phi_0) = 0$ and if $\mu(\Phi_0) > 0$, then $\sum_{n=-\infty}^{n=\infty} \mu(\Phi_0)$ diverges.

**8**. Problem 1(a)(b)(d) on page 450 in Pugh.

**Solution:** To be posted later.