Homework #9. Due Friday, November 9th, by 1pm

Reading:

1. For this homework assignment: Equicontinuity and compactness in function spaces (4.3 in Pugh, 7.6 in Rudin and Lectures 17 and 18).

2. For next week's classes: Stone-Weierstrass Theorem (4.4 in Pugh and 7.7 in Rudin)

Problems:

1. Let $a, b \in \mathbb{R}$ with $a < b$, and let $\{f_n\}$ be a sequence of differentiable functions from $[a, b]$ to $\mathbb{R}$. Suppose that both the sequences $\{f_n\}$ and $\{f_n'\}$ are uniformly bounded. Prove that the sequence $\{f_n\}$ is equicontinuous (and hence has a uniformly convergent subsequence).

2. The goal of this problem is to show that the statement of the Arzela-Ascoli theorem may be false if the domain is not totally bounded.

(a) Consider functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \begin{cases} \frac{|x|}{n} & \text{if } |x| \leq n \\ 1 & \text{if } |x| > n \end{cases}$
Prove that the sequence $\{f_n\}$ is uniformly bounded and equicontinuous, but does not have a uniformly convergent subsequence. Deduce that Arzela-Ascoli Theorem does not hold for $X = \mathbb{R}$.

(b)* (bonus) Now let (X, d) be any unbounded metric space. Show that there exists a sequence of continuous functions $f_n : X \to \mathbb{R}$ which is uniformly bounded and equicontinuous, but does not have a uniformly convergent subsequence.

3*. Pugh, problem 9 on p. 264.

4*. Pugh, problem 14 on p. 264

5. Let (X, d) be a metric space, and let $(B(X), d_{unif})$ be the metric space of all bounded functions $f : X \to \mathbb{R}$ with uniform metric:

$$d_{unif}(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Let $\mathcal{F} \subseteq B(X)$. Let (P) be one of the three properties: pointwise bounded, uniformly bounded and equicontinuous. Prove that if $\mathcal{F}$ has (P), then its closure $\overline{\mathcal{F}}$ also has (P). (You need to give three different proofs, one for each property).

Hint for 2(b): You can construct such a sequence using functions of the form $f(x) = d(x, a)$ (for a fixed $a \in X$).

Hint for 3: The answer is very easy to state and slightly harder to prove. For the proof use the fact that $f(x) - f(y) = f_n(\frac{x}{n}) - f_n(\frac{y}{n})$ for all $x, y \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Hint for 4: Prove the backward direction by contrapositive. First prove that a metric space (X, d) is NOT chain connected if and only if there exist $\delta > 0$ and non-empty A and B such that $X = A \sqcup B$ and $d(a, b) \geq \delta$ for all $a \in A, b \in B$. Once you know that X has the latter property, it is easy to construct an equicontinuous sequence of functions which is bounded at one point and not bounded at some other point.