Homework #8. Due Thursday, November 1st, in class

Reading:

1. For this homework assignment: In Pugh: 4.1 and 4.7; in Rudin: 7.3-7.5; class notes from lectures 14-16.

2. For next week's classes: Equicontinuity and compactness function spaces (4.3 in Pugh and 7.6 in Rudin).

Note: In some problems you will need to use Theorem 1 on p. 213 from Pugh, which is a stronger version of Theorem 14.1 from class: Theorem 14.1 asserts that uniform convergence preserves global continuity. Theorem 1 asserts that uniform convergence preserves continuity at any given point. The proof of Theorem 1 is basically the same. Also, Pugh proves the theorem for the case $X = [a, b]$, but the proof generalizes to an arbitrary metric space X without any changes.

Problems:

1. Let X be a metric space and $\{f_n\}, f$ functions from X to $\mathbb{R}$. Suppose that $f_n \rightrightarrows f$ on X.

(i) Prove that if each f_n is bounded, then f is bounded.

(ii) Assume that f is bounded. Prove that there exists $M \in \mathbb{N}$ and $C \in \mathbb{R}$ such that $|f_n(x)| \leq C$ for all $n \geq M$ and $x \in X$. In other words, prove that the sequence $\{f_n\}$ becomes uniformly bounded after we remove the first few terms at the beginning.

2. Problem 5 on p. 263 in Pugh (see Exercise 3.36 for the definition of jump and removable discontinuities). A clarification on the statement: in each part of the problem you are given some property (P) of functions; the question is the following: if each f_n has property (P), is it always true that the limiting function f also has (P). In part (e) countable should mean 'infinite countable'.

3. For each $\alpha \in \mathbb{R}$ define the function $I_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$I_\alpha(x) = \begin{cases} 0 & \text{if } x < \alpha \\ 1 & \text{if } x \geq \alpha \end{cases}$$

Now let $S = \{s_1, s_2, \ldots\}$ be a countable infinite subset of $\mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} \frac{I_{s_n}(x)}{2^n}$. Prove that

(a) the series always converges (so that f is indeed defined on $\mathbb{R}$),

(b) f is increasing (that is, $x < y$ implies $f(x) \leq f(y)$), and

(c)* f is continuous at $x \iff x \notin S$.

4. Problem 7.3:15 from Bergman's supplement to Rudin (page 79), see
`http://math.berkeley.edu/~gbergman/ug.hndts/m104_Rudin_exs.pdf`

You can assume that the functions are real-valued (not complex-valued); also J denotes the natural numbers.

Hint for 3(c): Use Weierstrass M-test (combined with Lemma 16.1). To prove that f is discontinuous at every $x \in S$ write $f = g + h$ where g is continuous at x and h is discontinuous at x (you would need different decompositions for different x).