Homework #7. Due Monday, October 29th, by 1pm (in my mailbox)

Reading:

1. For this homework assignment. Completions of metric spaces (2.10 in Pugh, class lecture 13). Uniform convergence (4.1 in Pugh, up to p.214 including the Example on p.214, 7.1-7.3 in Rudin, class lecture 14).
2. For next week's classes: More on uniform convergence (the rest of 4.1 in Pugh, 7.3-7.5 in Rudin). Construction of a continuous nowhere differentiable function (4.7 in Pugh and Theorem 7.18 on p.154 in Rudin).

Problems:

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

1. Let X be a metric space and Y a subset of X.

(a) Prove that if X is complete and Y is closed in X, then Y is complete.
(b) Prove that if Y is complete, then Y is closed in X.

Recall that we proved the analogous statements with 'complete' replaced by 'sequentially compact' (Theorem 9.2 and Theorem 8.1, respectively).

2. This problem describes a fancy way to show that closed bounded intervals in $\mathbb{R}$ are connected. A metric space (X, d) is called *chain-connected* if for any $x, y \in X$ and $\delta > 0$ there exists a finite sequence $x_0, x_1, \ldots, x_n$ of points of X such that $x_0 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \delta$ for all i.

(a)* Let X be metric space which is compact and chain-connected. Prove that X is connected.
(b) Prove that a closed bounded interval $[a, b] \subseteq \mathbb{R}$ is chain-connected and deduce from (a) that $[a, b]$ is connected.

3*. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

(a) Assume that f' is bounded, that is, there exists $M \in \mathbb{R}$ such that $|f'(x)| \le M$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.
(b) Now assume that $f'(x) \to \infty$ as $x \to \infty$. Prove that f is not uniformly continuous.

4. The goal of this problem is to fill in the details of the construction of the completion of a metric space discussed in Lecture 13. Part (a) below is Claim 1 from class; (b) and (c) form Claim 2 from class, and (d) is Claim 3 from class.

We start by recalling the notations introduced in class. Let (X, d) be a metric space. Let $\Omega = \Omega(X)$ be the set of all Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X$ for each n. Define the relation $\sim$ on Ω by setting

$$\{x_n\} \sim \{y_n\} \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

(a) Prove that $\sim$ is an equivalence relation.

Now let $\widehat{X} = \Omega/\sim$, the set of equivalence classes with respect to $\sim$. The equivalence class of a sequence $\{x_n\}$ will be denoted by $[x_n]$. For instance, $[\frac{1}{n}] = [\frac{1}{n^2}]$ since the sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$ are equivalent. Given an element $x \in X$, we will denote by $[x] \in \widehat{X}$ the equivalence class of the constant sequence all of whose elements are equal to x.

Now define the function $D : \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0}$ by setting

$$D([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n) \qquad (***)$$

(b)* Prove that the limit on the right-hand side of (***) always exists and that the function D is well-defined (that is, if $[x_n] = [x'_n]$ and $[y_n] = [y'_n]$, then $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$).

(c) Prove that $(\widehat{X}, D)$ is a metric space

(d) Consider the map $\iota : X \to \widehat{X}$ given by $\iota(x) = [x]$ (that is, ι sends each x to the equivalence class of the corresponding constant sequence). Prove that ι is injective and $D(\iota(x), \iota(y)) = d(x, y)$ for all $x, y \in X$. This implies that (X, d) is isometric to the metric space $(\iota(X), D)$ (so identifying X with $\iota(X)$, we can think of X as a subset of $(\widehat{X}, D)$).

5. Consider functions $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{nx+1}$. Let $0 \leq a \leq b$ be real numbers. Prove that $\{f_n\}$ converges uniformly on $[a, b] \iff a > 0$ or $a = b = 0$.

6. Let X be a set, (Y, d) a metric space. Let $\{f_n : X \to Y\}$ be a sequence of functions, and let $f : X \to Y$ be a function.

(i) Define what it should mean for $\{f_n\}$ to converge to f uniformly and what it should mean for $\{f_n\}$ to be uniformly Cauchy (in class we gave both definitions in the case $Y = \mathbb{R}$, but there is a natural way to extend them to arbitrary Y).

(ii) Theorem 14.2 from class asserts that in the case $Y = \mathbb{R}$, a sequence $\{f_n\}$ is uniformly convergent if and only if it is uniformly Cauchy. Find a natural necessary and sufficient condition on Y under which this equivalence remain true (the answer will not depend on X as long as $X \neq \emptyset$). You do not need to write down the full proof – just state the condition and where it arises in the proof.

Hint for 2(a): Assume that X is disconnected, and use Problem 2 from HW#6 and uniform continuity to reach a contradiction.

Hint for 3: Use the mean-value theorem.

Hint for #4(b): For the existence of the limit prove that the sequence $\{d(x_n, y_n)\}$ is Cauchy using the inequality $d(x, w) \leq d(x, y) + d(y, z) + d(z, w)$.