Homework #5. Due by 1pm on Friday, October 12th, in my mailbox

Reading:

1. For this homework assignment: read 2.2 and 2.3 again (there are some parts of those sections which we did not really emphasize before and which are relevant for this assignment)
2. For next week's class: read about connectedness (2.5 in both Pugh and Rudin)

Problems:

Note on hints: Most hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which a hint is available are marked with *.

1. Let A be a non-empty bounded above subset of $\mathbb{R}$. Prove that $\sup(A) \in \overline{A}$ directly from the definition of a contact point and the definition of supremum. You should give a short clean argument.

2. Prove Lemma 10.6 from class: if $\{x_n\}$ is a Cauchy sequence in some metric space X, and $\{x_n\}$ contains a convergent (in X) subsequence, then $\{x_n\}$ converges in X. You may use the fact that every metric space X is a subset of some complete metric space Y (we will prove this in Lecture 13). **Hint:** Start with the case $X = \mathbb{Q}$, $Y = \mathbb{R}$; then generalize to arbitrary X.

3. Let Z be a metric space and let Y be a dense subset of Z. Suppose that every Cauchy sequence in Y converges in Z. Prove that Z is complete.

4. Let X be a set, and let d_1 and d_2 be two different metrics on X. Given $x \in X$ and $\varepsilon > 0$, define $N^1_\varepsilon(x) = \{y \in X : d_1(y,x) < \varepsilon\}$, the ε-neighborhood of x with respect to d_1, and similarly define $N^2_\varepsilon(x) = \{y \in X : d_2(y,x) < \varepsilon\}$. We will say that d_1 and d_2 are *topologically equivalent* if a subset S of X is open with respect to d_1 $\iff$ it is open with respect to d_2. (**Note:** for brevity, if d is a metric on X, we will say that S is d-open if S is open as a subset of the metric space (X, d)).

(a) Prove that d_1 and d_2 are topologically equivalent if and only if for every $\varepsilon > 0$ and every $x \in X$ there exist $\delta_1, \delta_2 > 0$ (depending on both ε and x) such that $N^1_{\delta_1}(x) \subseteq N^2_\varepsilon(x)$ and $N^2_{\delta_2}(x) \subseteq N^1_\varepsilon(x)$.

(b) Suppose that there exist real numbers $A, B > 0$ such that $d_1(x, y) \leq Ad_2(x, y)$ and $d_2(x, y) \leq Bd_1(x, y)$ for all $x, y \in X$. Use (a) to prove that d_1 and d_2 are topologically equivalent.

(c) Now use (b) to prove that the Euclidean and Manhattan metrics on $\mathbb{R}^n$ are topologically equivalent.

5.

(a) Theorem 40 in Pugh states the following: Let X, Y are metric metric spaces, assume that X is a compact, and assume that $f : X \to Y$ is continuous and bijective. Then $f^{-1} : Y \to X$ is also continuous. Give a short proof of this theorem by combining Corollary 7.2, Theorem 8.4, Theorem 9.2 and Theorem 11.1 from class (the respective references in Pugh are Theorem 11, equivalence of (i) and (iii), Theorem 26, Theorem 32 and Theorem 36).

(b)* Use (a) to show that there exist metric spaces X and Y and a function $f : X \to Y$ such that f is continuous and bijective, but $f^{-1} : Y \to X$ is not continuous (so that the assumption that X is compact in (a) is essential).

6. Read (and understand) the proof of Theorem 42 in Pugh which asserts that if $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

Hint for 3: Show that for any sequence $\{z_n\}$ in Z there is a sequence $\{y_n\}$ in Y such that $d(y_n, z_n) \to 0$ as $n \to \infty$; then show that if $\{z_n\}$ is Cauchy, then $\{y_n\}$ is also Cauchy.

Hint for 5(b): Use a result from one of the previous homeworks.