Homework #4. Due Thursday, September 27th, in class

Reading:

1. For this homework assignment: In Pugh: 2.2, 2.3, the beginning of 2.4 and 2.7; in Rudin: 2.3 and 4.2.
2. For next week's classes: We will continue talking about compactness and its relation with completeness and continuity. The relevant sections are still 2.4 and 2.7 in Pugh and 2.3 in Rudin. I may send more detailed suggestions over the weekend.

Problems:

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which a hint is available are marked with *.

1. Let $f : A \to B$ be a function. Give a detailed proof of the following properties:

(a) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ for all $U, V \subseteq B$

(b) $f(f^{-1}(D)) \subseteq D$ for all $D \subseteq B$. Give an example showing that the inclusion may be strict.

(c) $f^{-1}(f(C)) \supseteq C$ for all $C \subseteq A$. Give an example showing that the inclusion may be strict.

2. Let $\{x_n\}$ be a sequence in a metric space (X, d), and let x be some element of X. Prove that the following conditions are equivalent:

(i) some subsequence of $\{x_n\}$ converges to x

(ii) for every $\varepsilon > 0$ there are infinitely many n for which $x_n \in N_\varepsilon(x)$.

When proving the implication (ii)$\Rightarrow$(i) make it clear how you use that $x_n \in N_\varepsilon(x)$ for infinitely many n (and not just for some n).

3. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Prove that K is covering compact in two different ways:

(i) by showing that K is closed and bounded as a subset of $\mathbb{R}$ (We will prove in class next week that a subset of $\mathbb{R}$ is covering compact if and only if it is closed and bounded).

(ii)* directly from definition of covering compactness.

4*. Let X be a metric space. Prove that X is covering compact $\iff$ X satisfies the following property:

> Let $\{K_\alpha\}$ be any collection of closed subsets of X such that for any finite subcollection $K_{\alpha_1}, \ldots, K_{\alpha_n}$, the intersection $K_{\alpha_1} \cap \ldots \cap K_{\alpha_n}$ is non-empty. Then the intersection of all sets in $\{K_\alpha\}$ is non-empty.

5*. Let $X = C[a,b]$, considered as a metric space with uniform metric d_{unif} (as defined in Problem 4 in HW#2). Prove that the set $B_1(\mathbf{0})$, the closed ball of radius 1 centered at $\mathbf{0}$ in X, is not sequentially compact. Here $\mathbf{0}$ is the function which is identically 0.

Recall that the notion of an *ultrametric* metric space was introduced in HW#3.7. The following problem gives an interesting example of an ultrametric metric space.

6. Let p be a fixed prime number. Define the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ as follows: given a nonzero $x \in \mathbb{Q}$, we can write $x = p^a \frac{c}{d}$ for some $a, c, d \in \mathbb{Z}$ where c and d are not divisible by p . Define $|x|_p = p^{-a}$ (note that the above representation is not unique, but it is easy to see that a is uniquely determined by x). For instance,

$$\left|\frac{9}{20}\right|_p = \begin{cases} \frac{1}{9} & \text{if } p = 3 \\ 4 & \text{if } p = 2 \\ 5 & \text{if } p = 5 \\ 1 & \text{for any other } p. \end{cases}$$

Also define $|0|_p = 0$. Now define the function $d_p : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}_{\geq 0}$ by $d_p(x,y) = |y - x|_p$.

(a) Prove that $(\mathbb{Q}, d_p)$ is an ultrametric space. (Note: the completion of this metric space is usually denoted by $\mathbb{Q}_p$ is called *p-adic numbers*).

(b) Describe explicitly the set $N_1(0)$ (the open ball of radius 1 centered at 0) in $(\mathbb{Q}, d_p)$.

(c) Let d be the standard metric on $\mathbb{Q}$ (that is, $d(x,y) = |y - x|$ where $|\cdot|$ is the usual absolute value). Give examples of sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{Q}$ such that

  (i) $x_n \to 0$ in $(\mathbb{Q}, d_p)$ but $\{x_n\}$ is unbounded as a sequence in $(\mathbb{Q}, d)$

(ii) $y_n \to 0$ in $(\mathbb{Q}, d)$ but $\{y_n\}$ is unbounded as a sequence in $(\mathbb{Q}, d_p)$

7. Let X be an arbitrary metric space and $f : X \to \mathbb{R}$ a continuous function (where $\mathbb{R}$ is equipped with standard metric).

(i) Prove that the sets $\{x \in X : f(x) > 0\}$ and $\{x \in X : f(x) < 0\}$ are open and the set $\{x \in X : f(x) = 0\}$ is closed

(ii) Prove that if $g : X \to \mathbb{R}$ is another continuous function, then the set $\{x \in X : f(x) = g(x)\}$ is closed

Hint for #3(ii): Let $\{U_\alpha\}$ be any open cover of K. One of the U_α must contain 0. What can you say about that U_α?

Hint for #4: There is a natural bijection between open covers of X and collections of closed subsets with empty intersection (here we consider open covers in the special case $S = X$).

Hint for #5: Use a result from one of the previous homeworks.