Homework #3. Due Thursday, September 20th, in class

Reading:

1. For this homework assignment: In Pugh: 2.1, 2.2 and parts of 2.3; in Rudin: 2.2 and 4.2 (pp. 85-89)
2. For next week's classes: We will continue talking about continuity. One thing we will definitely discuss is topological description of continuity (see pp. 71-73 in Pugh or Theorem 4.8 in Rudin). Then we will move on to compactness (2.4 and 2.7 in Pugh and 2.2 in Rudin). There are two equivalent definitions of compactness for metric spaces – "covering compactness" in the terminology of Pugh (this is also the right definition for the more general class of topological spaces and usually called just compactness) and sequential compactness (which in general is weaker than covering compactness, but is equivailent in the case of metric spaces). Rudin introduces compactness as covering compactness and barely mentions sequential compactness. Pugh starts with sequential compactness and uses it proves to prove all the main compactness results in 2.4. He introduces covering compactness much later, in 2.7. I plan to start with sequential compactness because it is easier to comprehend, but I will also introduce covering compactness before finishing 2.4.

Problems:

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

1. Given a metric space (X, d), a point $x \in X$ and $\varepsilon > 0$, define $B_\varepsilon(x) = \{y \in X : d(y, x) \leq \varepsilon\}$, called the *closed ball of radius ε centered at x.*

(a) Prove that $B_\varepsilon(x)$ is always a closed subset of X.

(b) Deduce from (a) that $\overline{N_\varepsilon(x)} \subseteq B_\varepsilon(x)$, that is, the closure of the open ball of radius ε centered at x is contained in the respective closed ball.

(c) Is it always true that $\overline{N_\varepsilon(x)} = B_\varepsilon(x)$? Prove or give a counterexample.

2*. Let X be metric space, and let $Z \subseteq Y$ be subsets of X. Prove that Z is closed as a subset of $Y \iff Z = Y \cap K$ for some closed subset K

of X. Deduce that if Z is closed in X, then Z is closed in Y. **Note:** The analogous result with closed sets replaced by open sets was proved in class (Theorem 5.2) and also appears as Theorem 13 on page 74 in Pugh or as Theorem 2.30 in Rudin.

3. Let (X, d) be a metric space and S a subset of X. Prove that the following three conditions are equivalent. The set S is called *bounded* if it satisfies either of those conditions:

(i) There exists $x \in X$ and $R \in \mathbb{R}$ such that $S \subseteq N_R(x)$.

(ii) For any $x \in X$ there exists $R \in \mathbb{R}$ such that $S \subseteq N_R(x)$.

(iii) The set $\{d(s,t) : s, t \in S\}$ is bounded above as a subset of $\mathbb{R}$.

Definition: Let (X, d) be a metric space and $\varepsilon > 0$. A subset S of X is called an *ε-net* if for any $x \in X$ there exists $s \in S$ such that $d(x, s) < \varepsilon$. In other words, S is an ε-net if X is the union of open balls of radius ε centered at elements of S.

4*. Let S be a subset of a metric space (X, d). Prove that the following are equivalent:

(i) The closure of S is the entire X;

(ii) $U \cap S \neq \emptyset$ for any non-empty open subset U of X;

(iii) S is an ε-net for every $\varepsilon > 0$.

The subset S is called *dense* (in X) if it satisfies these equivalent conditions.

5. Let X be any set with discrete metric ($d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$), and let Y be an arbitrary metric space.

(a) Let $\{x_n\}$ be a sequence in X. Prove that $\{x_n\}$ converges if and only if it is eventually constant, that is, there exists $M \in \mathbb{N}$ and $x \in X$ such that $x_n = x$ for all $n \geq M$.

(b) Prove that any function $f : X \to Y$ is continuous in two different ways: first using sequential definition of continuity and then using the ε-δ definition.

6.

(a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function such that

$$d_Y(f(u), f(v)) \leq d_X(u, v) \text{ for all } u, v \in X.$$

Prove that f is continuous.

(b) Let (X, d) be a metric space, and fix $a \in X$. Use (a) to prove that the function $f : X \to \mathbb{R}$ (where $\mathbb{R}$ is equipped with the usual metric) given by $f(x) = d(a, x)$ is continuous. Warning: be careful with absolute values.

7. A metric space (X, d) is called **ultrametric** if for any $x, y, z \in X$ the following inequality holds:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

(Note that this inequality is much stronger than the triangle inequality). If X is any set and d is the discrete metric on X, then clearly (X, d) is ultrametric. A more interesting example of an ultrametric space will be given in the next homework.

Prove that properties (i) and (ii) below hold in any ultrametric space (X, d) (note that both properties are counter-intuitive since they are very far from being true in $\mathbb{R}$).

(i) Take any $x \in X$, $\varepsilon > 0$ and take any $y \in N_\varepsilon(x)$. Then $N_\varepsilon(y) = N_\varepsilon(x)$. This means that if we take an open ball of fixed radius around some point x, then for any other point y from that open ball, the open ball of the same radius, but now centered at y, coincides with the original ball. In other words, any point of an open ball happens to be its center.

(ii) Prove that a sequence $\{x_n\}$ in X is Cauchy $\iff$ for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $d(x_{n+1}, x_n) < \varepsilon$ for all $n \geq M$. **Note:** The forward implication holds in any metric space. The definition of a Cauchy sequence is given on page 77 in Pugh. We will talk about Cauchy sequences later in class, but to solve this problem you do not need to know anything about them except the definition.

Hint for #2: Do not try to imitate the proof for the open set case (this is not impossible, but definitely not very natural). Instead use the inheritance property for open sets to prove the corresponding result for closed sets.

Hint for #4: Prove that negations of (i), (ii) and (iii) are equivalent to each other