Homework #2. Due Thursday, September 13th, in class

Reading:

1. For this homework assignment: In Pugh: 1.4, 2.1 and parts of 2.3 and 2.6; in Rudin: 2.1 and 2.2.
2. For next week's classes: TBA later

Problems:

Note on hints: Most hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

Definition: Two sets X and Y are said to have the same cardinality if there is a bijection from X to Y.

1. Let A be an uncountable set and B a countable subset of A.

(a) Prove that $A \setminus B$ is uncountable.

(b)* Prove that A and $A \setminus B$ have the same cardinality.

2*. Let X and Y be any sets, and define X^Y to be the set of all functions $f : Y \to X$. Prove that if $|X| \geq 2$, then Y and X^Y do not have the same cardinality.

3. A real number α is called algebraic if α is a root of a (nonzero) polynomial with **integer** coefficients, that is, if there exist integers $c_0, \ldots, c_n$, not all 0 such that $\sum\limits_{k=0}^{n} c_k \alpha^k = 0$. Note that all rational numbers are algebraic (if $\alpha = \frac{p}{q}$, then α is a root of the polynomial $qx - p$), but many irrational numbers are algebraic as well (e.g. $\sqrt{2}$ is algebraic as $\sqrt{2}$ is a root of $x^2 - 2$). Prove that the set of all algebraic numbers is countable.

If you need a hint, see Problem 1.39 in Pugh or 2.2 in Rudin or Problem 6 in

`http://people.virginia.edu/~mve2x/3000_Spring2018/homework11.pdf`

4. Let $a \leq b$ be real numbers and $X = C[a, b]$, the set of all continuous functions from $[a, b]$ to $\mathbb{R}$. Define the functions $d_{unif} : X \times X \to \mathbb{R}_{\geq 0}$ and

$d_{int} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$d_{unif}(f,g) = \max_{x \in X} |f(x) - g(x)| \text{ and } d_{int}(f,g) = \int\limits_a^b |f(x) - g(x)|\, dx.$$

(a) Prove that (X, d_{unif}) is a metric space (the metric d_{unif} is called the **uniform metric**)

(b) (practice) Prove that (X, d_{int}) is a metric space (the metric d_{int} is called the **integral metric**)

5. Let (X, d) be a metric space and S is a subset of X. Prove that S is open $\iff$ S is the union of some collection of open balls (which could be centered at different points).

6. Let (X, d) be a metric space and S a subset of X.

(i) Recall from Lecture 4 that a point $x \in X$ is called a *contact point of S* if $N_\varepsilon(x) \cap S \neq \emptyset$ for all $\varepsilon > 0$

(ii) A point $x \in X$ is called an *interior point of S* if there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$

The set of all contact points of S is denoted by $\overline{S}$ and the set of all interior points of S is denoted by S^o.

(a*) Prove that the set S^o is always open.

(b) Let $x \in S$. Prove that x is a contact point of S $\iff$ x is not an interior point of $X \setminus S$.

(c) Use (a) and (b) to prove that the set $\overline{S}$ is always closed (using the definition of a closed set given in class).

Note: Be aware of the following differences in terminology. **Contact points** are called **limit points** in Pugh (this is not obvious from the definition on page 66, but the proof is relatively simple). Rudin also uses the notion of a limit point, but it means something different (Rudin's use of the notion of a limit point is more common). Limit points in the sense of Rudin are called **cluster points** in Pugh.

The definitions of closure and closed set in Rudin and Pugh are different from the definitions that (will be) given in class in Lecture 4. Even though these definitions are equivalent to the ones from class, you have to use the definitions from class to solve this problem; otherwise, you will be trivializing some parts of the problem. For instance, 6(c) is automatically true according to the definition in Pugh.

7*. Let $X = C[a, b]$ and $d = d_{unif}$ (as defined in Problem 4). Find an (infinite) sequence $f_1, f_2, \ldots$ of elements of X such that $d(f_i, f_j) = 1$ for all $i \neq j$.

Hint for 1(b): Choose any countably infinite subset C of $A \setminus B$ and then use things proved in class to show that the identity map $f : (A \setminus B) \setminus C \to (A \setminus B) \setminus C$ can be extended to a bijection from $A \setminus B$ and A. Draw a picture!

Hint for 2: Note that if $X = \{0, 1\}$ and $Y = \mathbb{N}$, then X^Y is precisely the set of infinite sequences of 0 and 1, so the assertion of the problem in this special case holds by Theorem 3.6. To prove the general case imitate the proof of Theorem 3.6. See also Problem 1.38 in Pugh.

Hint for 6(a): Use the fact that open balls are open sets.

Hint for 7: As observed in class, if we replace $C[a,b]$ by $B[a,b]$, the set of all bounded functions from $[a,b]$ to $\mathbb{R}$ and define the metric d on $B[a,b]$ by $d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$, then the analogous question would have a very simple answer, e.g. we could let $f_n = I_{1/n}$ where I_c (for a fixed $c \in \mathbb{R}$) is the function defined by $I_c(c) = 1$ and $I_c(x) = 0$ for $x \neq c$. To solve Problem 7 think of a suitable way to approximate I_c by a continuous function.