Homework #11. Due on Thursday, Dec 6, in class

Reading:

1. For this homework assignment: Sections 28 and 29.1 in Kolmogorov-Fomin (class handout), class notes (Lectures 23-25), parts of Pugh (see below);

2. For next week's classes: Sections 29 and 30 in Kolmogorov-Fomin and 6.6 in Pugh. Note that Pugh's approach to Lebesgue integration is very different to the one we are discussing in class, but is also very interesting and geometrically intuitive, so make sure to read that section.

Problems:

0. Read about the Riemann/Darboux integral in Pugh (starting on p. 264 in Pugh until at least the end of p. 267). The definitions of the Riemann and Darboux integrals are different, but they lead to equivalent notions of integrability and the same value of the integral (Theorem 20 on p. 267).

1. Let X be a metric space and $A \subseteq X$. Then

- A is a called an F_σ-set if A is countable union of closed sets.
- A is a called a G_δ-set if A a countable intersection of open sets.

(i) Prove that A is an F_σ-set $\iff$ $X \setminus A$ is a G_δ-set.

(ii) Prove that the collection of all F_σ-sets in X is closed under countable unions and the collection of all G_δ-sets in X is closed under countable intersections.

(iii)* Let $X = \mathbb{R}$ (with standard metric). Prove that every open subset of X is an F_σ-set and every closed subset of X is a G_δ-set.

2. Let $D : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function (defined by $D(x) = 1$ if $x \in \mathbb{Q}$ and 0 if $x \notin \mathbb{Q}$), and let $a < b$ be real numbers.

(a) Prove that D is Lebesgue-integrable on $[a, b]$ and that $\int\limits_{[a,b]} D\, dm = 0$.

(b) Prove that D is not Riemann-integrable on $[a, b]$.

Make sure you did the reading in Problem 0 before solving this.

3*. Let $\{f_n : [a, b] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of measurable functions, and let A be the set of all $x \in [a, b]$ such that $\{f_n(x)\}$ converges. Prove that A is measurable.

4. Given a function $f : [a, b] \to \mathbb{R}$, let $\Gamma(f)$ be the graph of f, that is,

$$\Gamma(f) = \{(x, f(x)) : x \in [a, b]\} \subset \mathbb{R}^2.$$

Let m denote the Lebesgue measure on $\mathbb{R}^2$. In each part of this problem prove that $\Gamma(f)$ has measure 0 (each part is a generalization of the previous part, but please do not deduce (a) from (b) or (b) from (c) as there are easier constructions that work for (a) and (b))

(a) $f(x) = x$
(b)* f is an arbitrary continuous function
(c)* f is an arbitrary measurable function

Note: A standard way to prove that a set A has measure 0 is to show that for every $\varepsilon > 0$ one can find a set A_ε containing A with $m^*(A_\varepsilon) < \varepsilon$.

5. Let C be the standard Cantor set, and let $H : [0, 1] \to [0, 1]$ be the Cantor function AKA the Devil staircase function (see p.187 in Pugh).

(a) Prove that $m(H(C)) = 1$. This shows that a continuous function may send a set of measure zero to a set of positive measure.
(b)* Compute $\int\limits_{[0,1]} H \, dm$.
(c) (bonus) Modify the construction of H to show that for every $\varepsilon > 0$ there exists a **strictly** increasing continuous function $f_\varepsilon : [0, 1] \to [0, 1]$ such that $\mu(H_\varepsilon(C)) > 1 - \varepsilon$.

6. (practice) Kolmogorov-Fomin, Problem 6 after Section 28 (p.292)

7. Kolmogorov-Fomin, Problem 8 after Section 28 (p.292). **Note:** The functions $f_i^{(k)}$ are only defined for $1 \leq i \leq k$. It is probably useful to start by drawing the graphs of the first few functions in the sequence (say for $k = 1, 2, 3$).

Hint for 1(iii): Start by showing that every open interval is an F_σ-set. Once this is done, the rest follows by direct combination of previously known results.

Hint for 3: Use Cauchy criterion to express the set in question in terms of sets of the form $\{x : |f_n(x) - f_m(x)| < \frac{1}{k}\}$ using countable unions and countable intersections.

Hint for 4(b): Use uniform continuity to cover $\Gamma(f)$ by a finite union of rectangles whose total area is less than a given $\varepsilon > 0$.

Hint for 4(c): Let $\{s_n\}$ be the sequence of simple functions uniformly converging to f from the proof of Lemma 24.5 (by construction these s_n approximate f from below), and let s'_n be the analogous sequence approximating f from above (using the ceiling function instead of the floor function). How to use $\{s_n\}$ and $\{s'_n\}$ to find a set of small measure containing $\Gamma(f)$. Drawing the picture will almost certainly be helpful.

Hint for 5(b): Since C has measure 0, we have $\int_{[0,1]} H\,dm = \int_{[0,1]\setminus C} H\,dm$ (explain why). And the restriction of H to $[0,1]\setminus C$ is a simple function, so its integral can be computed just by summing a certain series.