Homework #10. Due Thursday, November 29th, in class

Reading:

1. For this homework assignment: Stone-Weierstrass Theorem (4.4 in Pugh and 7.7 in Rudin, class notes from Lectures 19-20), Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$ (Section 25 in Kolmogorov-Fomin, 6.1 and 6.2 in Pugh and class notes from Lectures 21-23). Note that in class we defined measurable sets as in KF; Pugh's definition is quite different (albeit equivalent)

2. For next week's classes: TBA

Problems:

1. Let $a < b$ be real numbers and let $\mathcal{P}_{even}[a,b] \subseteq C[a,b]$ be the set of all even polynomials (that is, polynomials which only involve even powers of x).

(a) Use Stone-Weierstrass Theorem to prove that $\mathcal{P}_{even}[a,b]$ is dense in $C[a,b] \iff 0 \notin (a,b)$.

(b)* (optional) Now prove the "$\Leftarrow$" direction in (a) using only Weierstrass Approximation Theorem (but not Stone-Weierstrass Theorem).

2.

(a)* Prove that the (direct) analogue of Weierstrass Approximation Theorem does not hold for $C(\mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$: Show that there exists $f \in C(\mathbb{R})$ which cannot be uniformly approximated by polynomials, that is, there is no sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $\mathbb{R}$.

(b)* Now prove that the following (weak) version of Weierstrass Approximation Theorem holds for $C(\mathbb{R})$: for any $f \in C(\mathbb{R})$ there exists a sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $[a,b]$ for any closed interval $[a,b]$ (of course, the point is that a single sequence will work for all intervals).

3. (practice) Let A_1, A_2, B_1 and B_2 be subsets of the same set. Prove that

(a) $(A_1 \cup A_2) \triangle (B_1 \cup B_2) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$

(b) $(A_1 \cap A_2) \triangle (B_1 \cap B_2) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$

Recall that both formulas were used in verification of properties of Lebesgue measure.

4. In all parts of this problem $X = \mathbb{R}$ or $\mathbb{R}^2$, and m denotes the Lebesgue measure on X.

(a)* Prove that every open subset of X is measurable. Deduce that every closed subset of X is measurable.

Now let $\Omega_0, \Omega_1, \Omega_2, \dots$ be the following collections of subsets of X. First define Ω_0 to be the set of all subsets of X which are either open or closed. For each $k \geq 1$ define Ω_k to be the set of all subsets which can be represented either as a countable union or a countable intersection of subsets from Ω_{k-1}.

(b) Deduce from (a) that each set in each Ω_k is measurable

(c) Assume that $X = \mathbb{R}$ and $S = \mathbb{Q}$. Does there exist $k \in \mathbb{N}$ such that $S \in \Omega_k$? If yes, what is the smallest such k?

(d) Same question as (c) for $S = \mathbb{R} \setminus \mathbb{Q}$

5.

(a) Let A be a countable subset of $\mathbb{R}$. Prove that A has measure zero (that is, A is measurable and $m(A) = 0$).

(b) Prove that the (standard) Cantor set C has measure 0 (see p.105 in Pugh for the definition of the standard Cantor set).

6.

(a) Let A, B and C be subsets of the same set. Prove that

$$A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$$

(b) Now let $X = [0, 1]$ or $[0, 1]^2$. Let A be a subset of X, and suppose that for every $\varepsilon > 0$ there exists a measurable subset $B \subseteq X$ such that $m^*(A \triangle B) < \varepsilon$. Prove that A is measurable.

7. Problem 7 from Kolmogorov-Fomin (p. 268). Note that the hint given in KF is essentially a sketch of the solutions. The things you need to justify are

(a) $C = \cup_{n=-\infty}^{\infty} \Phi_n$

(b) $\Phi_n \cap \Phi_m = \emptyset$ if $n \neq m$

(c) Assume that Φ_0 is measurable. Then each Φ_n is measurable and $\mu(\Phi_n) = \mu(\Phi_0)$ for all $n \in \mathbb{Z}$

(d) the conclusion of (c) contradicts (33) in KF.

Remark: The Lebesgue measure on the circle C can be defined in exactly the same way as on $\mathbb{R}$ with the exception that we call a subset of C elementary if it is a finite union of arcs.

8. Problem 1(a)(b)(d) on page 450 in Pugh.

Hint for 1(b): WOLOG assume that $0 \leq a < b$. Start by showing that any continuous function in $g \in C[a,b]$ can be written as $g(x) = h(x^2)$ for some continuous function $h \in C[a^2, b^2]$.

Hint for 2(a): Use the fact that any non-constant polynomial $p(x)$ tends to $\pm\infty$ as $x \to \infty$.

Hint for 2(b): It is enough to prove the result for intervals of the form $[-k, k]$ for $k \in \mathbb{N}$ (why?). To construct a sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $[-k, k]$ for each k, apply Weierstrass Approximation Theorem on each interval and then use a diagonal-type argument.

Hint for 4(a): Show that any open subset of $\mathbb{R}^2$ can be written as a union of squares whose endpoints have rational coordinates.