

Homework #1. Due Friday, September 7th, by 1pm in my mailbox

Reading:

1. For this homework assignment: In Pugh: Section 1.2 (cuts); in Rudin: Chapter 1
2. Before the class on Tue, Sep 1: Basics on cardinality of sets (1.4 in Pugh or 2.1 in Rudin)
Before the class on Thu, Sep 3: Metric spaces, open and closed sets (2.1 and parts of 2.3 and 2.6 in Pugh and 2.2 in Rudin).

Problems:

1. Prove that \mathbb{C} (complex numbers) cannot be made an ordered field (no matter how the order relation $<$ is defined). **Note:** in class we proved this for lexicographic order; the general proof is not much more complicated.
2. Let S be a subset of \mathbb{R} bounded above, and let $-S = \{-x : x \in S\}$. Prove that $-S$ is bounded below and $\inf(-S) = -\sup(S)$.
3. Let S be an ordered set and A and B subsets of S such that
 - (i) $a \leq b$ for any $a \in A$ and $b \in B$;
 - (ii) $\sup(A)$ and $\inf(B)$ exist in S .

Prove that $\sup(A) \leq \inf(B)$ (there is a hint at the end of the assignment).

4. Let $(X, <)$ be an ordered set. A subset S of X will be called *basic* if the following is true: if $s \in S$ and $x \in X$ and $x < s$, then $x \in S$ (that is, for each element $s \in S$, all elements of X smaller than s also lie in S). For instance, any interval of the form $(-\infty, \alpha)$ or $(-\infty, \alpha]$ is a basic subset of \mathbb{R} .

- (a) Given $a \in X$, define $D_{<}(a) = \{x \in X : x < a\}$ and $D_{\leq}(a) = \{x \in X : x \leq a\}$. Prove that the sets $D_{<}(a)$ and $D_{\leq}(a)$ are basic.
- (b) Let $S \subseteq X$ be a basic set, and suppose that $\sup(S)$ exists (in X ; we do not assume that $\sup(S) \in S$). Prove that $S = D_{<}(\sup(S))$ or $S = D_{\leq}(\sup(S))$. **Hint:** It is clear that $S \subseteq D_{\leq}(\sup(S))$ for any set S such that $\sup(S)$ exists; suppose now that S is also basic and that S is different from $D_{<}(\sup(S))$ and $D_{\leq}(\sup(S))$. This implies that there exists $y \in X$ such that $y < \sup(S)$ and $y \notin S$ (explain why). Now

use the fact that S is basic to get a contradiction with the definition of $\sup(S)$.

In parts (c) and (d) of this problem $X = \mathbb{C}$ and $<$ is the lexicographic order relation on \mathbb{C} defined as follows: $a + bi < c + di$ (with $a, b, c, d \in \mathbb{R}$) $\iff a < c$ or ($a = c$ and $b < d$)

- (c) For each $z = u + vi \in \mathbb{C}$ describe explicitly the set $D_{<}(z)$
- (d) Find an (explicit) example of a non-empty subset S of \mathbb{C} which is bounded above but such that $\sup(S)$ does not exist (and prove that your subset has the required property). **Hint:** According to part (b) it is enough to find a set S which is basic, bounded above and such that S is not equal to $D_{<}(z)$ or $D_{\leq}(z)$ for any $z \in \mathbb{C}$ (the latter condition can be verified using the answer in (c)).

5. Give a detailed and rigorous proof of the fact that

$$\lim_{n \rightarrow \infty} \frac{2n + 3}{3n + 4} = \frac{2}{3}$$

directly from the definition of limit of a sequence.

6. Let $S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$. Prove by contradiction that S does not have a supremum in \mathbb{Q} in two ways:

- (a) Assuming the existence of \mathbb{R} and the fact that \mathbb{Q} is dense in \mathbb{R} , that is, every non-empty open interval in \mathbb{R} contains a rational number
- (b) (bonus) using just \mathbb{Q} . For this problem use the description of S which does not involve $\sqrt{2}$, e.g. $S = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$.

7. Deduce the Intermediate Value Theorem and Extreme Value Theorems directly from the following four results (which will be proved later in the course):

- (1) Let $I = [a, b]$ be a closed bounded interval in \mathbb{R} , and consider I as a metric space with the standard metric ($d(x, y) = |x - y|$). Then I is compact and connected.
- (2) Let $S \subseteq \mathbb{R}$ be a subset which is both compact and connected (again with respect to the standard metric). Then $S = \emptyset$ or $S = [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$.

- (3) Let X and Y be metric spaces and $f : X \rightarrow Y$ be a continuous function. If X is connected, then $f(X)$ is connected (as usual $f(X) = \{f(x) : x \in X\}$ is the image (=range) of f).
- (4) Let X and Y be metric spaces and $f : X \rightarrow Y$ be a continuous function. If X is compact, then $f(X)$ is compact.

Note: The definitions of compactness and connectedness for metric spaces are given in Chapter 2 of both Pugh and Rudin, but they are not needed for this problem (all you need to know is that these are certain properties of metric spaces).

Hint for Problem 3: First show that $\sup A \leq b$ for every $b \in B$ – this follows directly from the definition of supremum.