Math 5310. Intro to Real Analysis. Fall 2014. Final Exam.
Tuesday, December 16th, 9am-12pm

Directions: No books, notes, calculators, laptops, PDAs, cellphones, web appliances, or similar aids are allowed. All work must be your individual efforts.

- Show all your work and justify all statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

problem	1	2	3	4	5	6	7	total
points	12	8	10	10	16	8	8	60+12
score								

Scoring system: Exam consists of **7** problems, worth 72 points combined. Your total will be simply the sum of scores on individual problems (no weights this time). Thus, the maximal possible total is 72, but the score of 60 counts as 100%.

Problem 1: (12 pts) Prove that closed bounded intervals in $\mathbb{R}$ are compact (directly from the definition of compactness).

Problem 2: (8 pts) Let $f : [a, b] \to \mathbb{R}$ be a measurable function. According to the definition from class, this means that for any $c \in \mathbb{R}$, the set

$$f_{<c} = \{x \in [a, b] : f(x) < c\}$$

is measurable. Prove the following assertions directly from this definition (without using any results about measurable functions). You can use any results about measurable sets proved in class or in the homework.

(i) Prove that for any $c \in \mathbb{R}$, the set $f_{\leq c} = \{x \in [a, b] : f(x) \leq c\}$ is measurable.

(ii) Prove that for any $c < d$ in $\mathbb{R}$, the set $\{x \in [a, b] : c < f(x) < d\} = f^{-1}((c, d))$ is measurable.

(iii) Prove that for any open set $U \subseteq \mathbb{R}$, the set $f^{-1}(U)$ is measurable.

Problem 3: (a) (3 pts) State the Arzela-Ascoli Theorem.
(b) (7 pts) Let $\{f_n : [0, 1] \to \mathbb{R}\}$ be a **uniformly bounded** sequence of continuous functions, and define $F_n : [0, 1] \to \mathbb{R}$ by $F_n(x) = \int\limits_0^x f_n(t)\, dt$. Prove that the sequence $\{F_n\}$ has a uniformly convergent subsequence. In this problem you are not allowed to refer to results from homework.

Problem 4: (10 pts) Let (X, d) be a metric space, let $\{f_n : X \to \mathbb{R}\}$ be an **equicontinuous** sequence of functions, and let

$$C = \{x \in X : \{f_n(x)\} \text{ converges }\}.$$

Prove that C is closed.
Hint: You can prove directly that C is closed or that its complement is open – both methods will take roughly the same amount of work. In either case you will probably need to use an $\frac{\varepsilon}{3}$-trick.

Problem 5:

(a) (4 pts) State the Stone-Weierstrass Theorem and define all the relevant concepts (algebra, 'separate points' etc.)

(b) (4 pts) Let $X = [0, 1] \times [0, 1] = \{(x, y) : 0 \leq x, y \leq 1\}$, a unit square in $\mathbb{R}^2$ with Euclidean metric, and let

$$A = \left\{ f : X \to \mathbb{R} \text{ s.t. } f(x, y) = \sum_{i,j=0}^{n} a_{ij} x^i y^j : a_{i,j} \in \mathbb{R} \text{ for all } (x, y) \in X \right\},$$

the set of all polynomials (with real coefficients) in two variables. Prove that A is dense in $C(X)$.

(c) (5 pts) Now let $X = [0,1]$ and

$$A = \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{Q} \right\},$$

the set of all polynomials with RATIONAL coefficients. Prove that A is dense in $C(X)$.

Hint: Rephrase the desired conclusion 'A is dense in $C(X)$' in the form 'for any $\varepsilon > 0$ and $f \in C(X)$ there exists ...' You cannot solve this problem just by using Stone-Weierstrass theorem.

(d) (3 pts) Finally let $X = [0,1]$ and

$$A = \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{Z} \right\},$$

the set of all polynomials with INTEGER coefficients. Prove that A is NOT dense in $C(X)$.

Problem 6: For each of the following statements determine whether it is true (in general) or false (in some cases). If the statement is true, briefly explain why; if not, give a counterexample (and prove that it is a counterexample).

(a) (2 pts) If X is any metric space with $|X| \geq 2$, there exists a non-constant continuous function $f : \mathbb{R} \to X$.

(b) (2 pts) If X is any metric space with $|X| \geq 2$, there exists a non-constant continuous function $f : X \to \mathbb{R}$.

(c) (4 pts) If $X \subseteq [0,1]$ is a set of measure zero and $f : [0,1] \to [0,1]$ is a continuous function, then $f(X)$ has measure zero.

Problem 7: (8 pts) Prove that the assertion of the Arzela-Ascoli Theorem DOES hold for the non-compact space $X = (0,1)$ (the open interval $(0,1)$ with the standard metric).

Hint: There are two places in the proof of the Arzela-Ascoli Theorem where the assumption 'X is compact' is used. Show that in the case $X = (0,1)$ the argument can be adjusted in those two places (even though X is not compact) without affecting the rest of the proof. You do not need to give the full proof – just explain where the standard proof needs to be modified. (A couple of points will be given just for explaining how compactness is used in the proof of the Arzela-Ascoli Theorem).