Homework #9. Due Thursday, November 5th, in class

Reading:

1. For this homework assignment: 7.6 (Equicontinuous families of functions) + class notes (Lectures 17-18)

2. For next week's classes: Kolmogorov-Fomin 8.2 (contraction mappings and differential equations; handout was distributed in class on October 29th) and Rudin 7.7 (the Stone-Weierstrass Theorem); note that the main theorem of that section is Theorem 7.32.

Problems:

1. Let $a, b \in \mathbb{R}$ with $a < b$, and let $\{f_n\}$ be a sequence of differentiable functions from $[a, b]$ to $\mathbb{R}$. Suppose that both the sequences $\{f_n\}$ and $\{f_n'\}$ are uniformly bounded. Prove that $\{f_n\}$ has a subsequence which converges uniformly on $[a, b]$.

2*. Let X be a compact metric space, $(C(X), d_{unif})$ the space of continuous functions from X to $\mathbb{R}$ with uniform metric d_{unif} (given by $d_{unif}(f, g) = \max_{x\in X} |f(x) - g(x)|$). Prove that a subset $\mathcal{F}$ of $C(X)$ is compact (with respect to d_{unif}) $\iff$ $\mathcal{F}$ satisfies the following three conditions:

- (i) $\mathcal{F}$ is uniformly closed, that is, $\mathcal{F}$ is closed as a subset of $(C(X), d_{unif})$
- (ii) $\mathcal{F}$ is uniformly bounded (this is equivalent to saying that $\mathcal{F}$ is bounded as a subset of $(C(X), d_{unif})$)
- (iii) $\mathcal{F}$ is equicontinuous

3. Prove that the assertion of Arzela-Ascoli Theorem remains true for the non-compact metric space $X = (a, b)$ (where $a < b$ are real numbers). You just need to explain how the proof of Arzela-Ascoli Theorem given in class needs to be modified (in the place where compactness of X was used).

4.

- (a) Consider functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \begin{cases} \frac{|x|}{n} & \text{if } |x| \le n \\ 1 & \text{if } |x| > n \end{cases}$
  Prove that the sequence $\{f_n\}$ is uniformly bounded and equicontinuous, but does not have a uniformly convergent subsequence. Deduce that Arzela-Ascoli Theorem does not hold for $X = \mathbb{R}$.
- (b)* (bonus) Now let (X, d) be any unbounded metric space. Show that there exists a sequence of continuous functions $f_n : X \to \mathbb{R}$ which is uniformly bounded and equicontinuous, but does not have a uniformly convergent subsequence.

5. Problem 7.3:15 from Bergman's supplement (page 79), see
`http://math.berkeley.edu/~gbergman/ug.hndts/m104_Rudin_exs.pdf`

You can assume that the functions are real-valued (not complex-valued); also J denotes the natural numbers.

Hint for 2: For the forward direction, the main thing to prove is that $\mathcal{F}$ is equicontinuous. Assume that $\mathcal{F}$ is not equicontinuous, deduce that $\mathcal{F}$ contains a sequence with no equicontinuous subsequence (this is somewhat similar to the first step in the solution to Problem 6 in HW#5) and then use Theorem 7.24 from Rudin. For the backwards direction combine Arzela-Ascoli Theorem with the fact that $(C(X), d_{unif})$ is a complete metric space (Corollary 15.3 from class = Theorem 7.15 in Rudin).

Hint for 4(b): You can construct such a sequence using functions of the form $f(x) = d(x, a)$ (for a fixed $a \in X$).

Hint for 5(b): For the first part start by showing that F is always continuous at all points (p, s) with $s \neq 0$. Thus, you need to prove that $f_n \to f$ pointwise $\iff$ F is continuous at $(p, 0)$ for every $p \in X$.

For the second part show that if two points in X are close to each other, then their first coordinates are the same and the second coordinates are both close to 0; more formally, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d((p, s), (p', s')) < \delta$ implies $p = p'$ and $s < \varepsilon, s' < \varepsilon$. You will also need to use the Cauchy criterion.