Homework #7. Due Thursday, October 29th, in class

Reading:

1. For this homework assignment: 4.5-4.6 (Discontinuities, Monotonic Functions), 7.2-7.5 + class notes (Lectures 14-16)

2. For next week's classes: Section 7.6 (equicontinuous families of functions). I plan to follow Rudin quite closely in this section.

Note: In some problems you may want to use the following stronger version of Theorem 15.1 from class: *Let X be a metric space, and fix $a \in X$. Let $\{f_n\}_{n\in\mathbb{N}}, f$ be functions from X to $\mathbb{R}$, and suppose that $f_n \rightrightarrows f$ and each f_n is continuous at a. Then f is also continuous at a.*

(In class we showed that global continuity of each f_n implies global continuity of f, but we did not explicitly state that continuity of each f_n at a single point a forces f to be continuous at the same point; however, the proof given in class actually establishes the above stronger form of Theorem 15.1).

Problems:

1. Consider functions $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{nx+1}$. Let $0 \leq a \leq b$ be real numbers. Prove that $\{f_n\}$ converges uniformly on $[a,b] \iff a > 0$ or $a = b = 0$.

2*. Let X be a metric space and $\{f_n\}, f$ functions from X to $\mathbb{R}$. Suppose that $f_n \rightrightarrows f$ on X and each f_n is uniformly continuous. Prove that f is uniformly continuous. **Hint:** Imitate the proof of Theorem 15.1.

3. For each $\alpha \in \mathbb{R}$ define the function $I_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$I_\alpha(x) = \begin{cases} 0 & \text{if } x < \alpha \\ 1 & \text{if } x \geq \alpha \end{cases}$$

Now let $S = \{s_1, s_2, \ldots\}$ be a countable infinite subset of $\mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} \frac{I_{s_n}(x)}{2^n}$. Prove that

(a) the series always converges (so that f is indeed defined on $\mathbb{R}$),

(b) f is increasing (that is, $x < y$ implies $f(x) \leq f(y)$), and

(c)* f is continuous at $x \iff x \notin S$.

4*. Before reading this problem read sections 4.5 and 4.6 in Rudin. You may also want to look at

`http://people.virginia.edu/~mve2x/3310_Spring2015/lecture15.pdf`

Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Prove that the following are equivalent:

(i) f is continuous (on $[a,b]$)

(ii) $f([a,b]) = [f(a), f(b)]$

(iii) $f([a,b])$ is dense in $[f(a), f(b)]$

5. Give a short proof of Theorem 7.17 from Rudin under the additional assumption that each f_n' is continuous (naturally, you should not be repeating or imitating the proof in the general case given in Rudin). **Hint:** in the notations of Theorem 7.17, if we set $g(x) = \lim_{n\to\infty} f_n'(x)$, then g is a uniform limit of continuous functions, hence g is also continuous (Theorem 15.1). Now define $f : [a,b] \to \mathbb{R}$ by $f(x) = \int\limits_{x_0}^{x} g(t)\,dt + C$ for suitable C and prove that $f_n \rightrightarrows f$ using the Fundamental Theorem of Calculus (recall that $\int\limits_{c}^{d} g(t)\,dt$ in the case $c > d$ is defined as $-\int\limits_{d}^{c} g(t)\,dt$).

6. (practice) Let X be a any set.

(a) Let $B(X)$ be the set of all bounded functions $f : X \to \mathbb{R}$. Define the map $d : B(X) \times B(X) \to \mathbb{R}_{\geq 0}$ by

$$d(f,g) = \sup_{x\in X} |f(x) - g(x)|.$$

Prove that d is a metric on $B(X)$ which satisfies the following property: given $f_n, f \in B(X)$, we have $f_n \rightrightarrows f$ (on X) $\iff$ $d(f_n, f) \to 0$ as $n \to \infty$. Equivalently, $f_n \rightrightarrows f$ (on X) $\iff$ $f_n \to f$ in the metric space $(B(X), d)$.

Note: At the end of Lecture 14 we discussed the corresponding result for the metric space $C(X)$ (with uniform metric), where X is a compact metric space. Note that if X is compact, then $C(X) \subseteq B(X)$ by Theorem 4.16 in Rudin, and moreover, the above metric d restricted to $C(X)$ coincides with the uniform metric on $C(X)$.

(b) The goal of this part is to establish the analog of (a) with $B(X)$ replaced by $Func(X, \mathbb{R})$, the set of all functions from $X \to \mathbb{R}$.

Let $\Omega = Func(X, \mathbb{R})$, and define $D : \Omega \times \Omega \to \mathbb{R}_{\geq 0}$ by

$$D(f,g) = \sup_{x\in X} \min\{|f(x) - g(x)|, 1\}.$$

Equivalently,

$$D(f,g) = \begin{cases} \sup\limits_{x\in X} |f(x) - g(x)| & \text{if } |f(x) - g(x)| \leq 1 \text{ for all } x \in X \\ 1 & \text{otherwise} \end{cases}$$

Prove that D is a metric on Ω such that given $\{f_n\}, f \in \Omega$, we have $f_n \rightrightarrows f$ on X $\iff$ $f_n \to f$ in the metric space (Ω, D).

Hint for 2: Imitate the proof of Theorem 15.1.

Hint for 3(c): Use Weierstrass M-test (combined with Lemma 16.1). To prove that f is discontinuous at every $x \in S$ write $f = g + h$ where g is continuous at x and h is discontinuous at x (you would need different decompositions for different x).

Hint for 4: It is easy to show that (i)$\Rightarrow$ (ii)$\Rightarrow$ (iii). To prove that (iii)$\Rightarrow$(i) use the results about discontinuities of increasing functions proved in Rudin. Recall that we gave several (equivalent) definitions of a dense subset; your solution will be considerably simpler if you use the optimal (for the purposes of this problem) definition of density.